A New Outer Measure on the Space of Model-Free Generalized Processes

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Investigations in game-theoretic approach to model-free, financial models of continuous price paths culminated in Glenn Shafer and Vladimir Vovk publishing their book [3]. In their book Shafer and Vovk introduced a new notion – the notion of *instant enforcement*. But they did not characterize it using any outer measure, unlike what Vovk did in the case of sets of "typical" price paths. Properties which hold with *instant enforcement* are, roughly speaking, such properties (subsets of $[0, +\infty) \times \Omega$) consisting of pairs of $t \in [0, +\infty)$ (time) and $\omega \in \Omega$ (which may be interpreted as an elementary event or outcome of reality) that a trader (Skeptic) is able to become infinitely rich as soon as they cease to hold, see [3, Chapt. 14].

In the paper [2] we introduce an outer measure on the space $[0, +\infty) \times \Omega$, which assigns zero value exactly to those sets (properties) of pairs of time t and an elementary event ω, complements of which are instantly enforceable. We also introduce a slight modification of this measure (an open question is whether the introduced modification differs from the original measure) which allows us to establish Itô's isometry and Burkholder-Davis-Gundy inequalities (BDG inequalities in short) for this modification. BDG inequalities are standard tools in classical stochastic calculus to define a stochastic integral. They give bounds of expectation of the square (or other power) of such integral in terms of its quadratic variation (in financial models quadratic variation is associated with volatility). Fortunately, the quadratic variation of the stochastic integral may be expressed as an usual Riemann-Stieltjes integral. Such results were not present in Vovk or Shafer's works. A "weak" BDG inequality in a model-free setting, but quite different from ours (and only for $p = 2$) was established in [1]. A main novelty in our approach is that instead of working with (outer) expectation $\bar{\mathbb{E}}$ defined for variables $X : \Omega \to [-\infty, +\infty]$, as for example in [3, Sect. 13.3] or [1], we introduce an upper expectation $\overline{\overline{\mathbb{E}}}X$ (or cost of super-hedging or super-replication) of a generalized process $X : [0, +\infty) \times \Omega \to [-\infty, +\infty]$. Thus $\overline{\overline{\mathbb{E}}}$ is defined on the space of generalized processes not on the space of variables.

References

[1] D. Bartl, M. Kupper, and A. Neufeld. Stochastic integration and differential equations for typical paths. *Electron. J. Probab.*, 24(97):1–21, 2019. doi:10.1214/19-EJP343. URL https://projecteuclid.org/euclid.ejp/1568793790.

[2] R. Łochowski. BDG inequalities for model-free continuous price paths with instant enforcement. *Preprint arXiv:2109.07928*, 2022.

[3] G. Shafer and V. Volk. *Game-Theoretic Foundations for Probability and Finance*. Wiley Series in Probability and Statistics. Wiley, 2019.