Comonotone (Natural) Extension of Coherent Lower Probabilities

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Modelling the dependence structure between random variables is an important topic in probability theory that is related to the theory of Copulas and Sklar's Theorem [4]. In this paper we deal with a particular type of dependence, called *comonotonicity*, which refers to the extreme case of positive dependence, in the sense that there exists an increasing relationship between the random variables X and Y. In a finite framework, this property can be expressed in a number of equivalent ways: (i) $Y = h(X)$ for an increasing function h; (ii) the joint cumulative distribution function (cdf, for short) is the minimum of the marginals: $F_{X,Y}(x, y) = \min\{F_X(x), F_Y(y)\}$; or (iii) the support of (X, Y) is an increasing set, meaning that if (x_1, y_1) and (x_2, y_2) have strictly positive probability, either $x_1 \leq x_2$ and $y_1 \leq y_2$ or $x_1 \geq x_2$ and $y_1 \geq y_2$. An important property of comonotone random variables is related to the existence, construction and uniqueness of a *comonotone extension*: given two cdfs F_X and F_Y, there always exists a joint cdf $F_{X,Y}$ whose marginals are F_X and F_Y, that comonotone distribution can be easily built and it is unique. We refer to Dhaene et al. [1] for a survey about comonotonicity.

How to define comonotonicity for coherent lower probabilities in a finite framework has been analysed in [3]. There, a coherent lower probability $\underline{P}_{X,Y}$ was defined to be comonotone when its support, formed by the elements with strictly positive upper probability, is an increasing set. Besides analysing some equivalent representations of this property, it was also proven that under comonotonicity, the bivariate p-box $(\underline{F}_{X,Y}, \overline{F}_{X,Y})$ associated with $\underline{P}_{X,Y}$ can be expressed as $\underline{F}_{X,Y}(x, y) = \min\{\underline{F}_X(x), \underline{F}_Y(y)\}$ and $\overline{F}_{X,Y}(x, y) = \min\{\overline{F}_X(x), \overline{F}_Y(y)\}$, where $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ are the marginal p-boxes. However, it was shown that the converse does not hold in general.

In our previous paper [2] we analysed the problem of the existence, construction and uniqueness of a coherent lower probability $\underline{P}_{X,Y}$ whose marginal p-boxes $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ are given, called a *comonotone extension* of the marginal p-boxes. We proved that (i) a comonotone extension may not exist, but we characterised its existence; (ii) when it exists, we gave a constructive approach; (iii) when it exists, the comonotone extension may not be unique. The lack of uniqueness led us to investigate the *comonotone natural extension* of the marginal p-boxes, which is the smallest comonotone extension. Even if it may not exist, we also characterised its existence and we gave a constructive method for obtaining it.

In this contribution we address a more general problem where the marginal information is given in terms of coherent lower probabilities $\underline{P}_X$ and $\underline{P}_Y$ instead of univariate p-boxes. With this information, we analyse the same problem: the existence, construction and uniqueness of a comonotone lower probability $\underline{P}_{X,Y}$ whose marginals are $\underline{P}_X$ and $\underline{P}_Y$, called again the *comonotone extension* of $\underline{P}_X$ and $\underline{P}_Y$. Our results show that: (i) a comonotone extension may not exist, but we characterise its existence. Even if the necessary and sufficient conditions are a bit involved, checking whether $\underline{P}_X$ and $\underline{P}_Y$ satisfy them can be turned into a simple problem from graph theory; (ii) when it exists we give a constructive approach; and (iii) it may not be unique. Indeed, there may be infinitely many comonotone extensions. Due to the lack of uniqueness we explore again the existence of the *comonotone natural extension* of $\underline{P}_X$ and $\underline{P}_Y$, that is, the smallest comonotone extension. We characterise the existence of the comonotone natural extension of $\underline{P}_X$ and $\underline{P}_Y$, and we show that the conditions for the existence of the comonotone natural extension of $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ are sufficient for the existence of the comonotone natural extension of $\underline{P}_X$ and $\underline{P}_Y$, but not necessary. Finally, again we make use of graph theory tools to check whether the comonotone natural extension exists.

References

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