A Parametric Approach to the Estimation of Convex Risk Functionals Based on Wasserstein Distance*

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We study a class of convex risk functionals which arises naturally in the context of mathematical finance and actuarial science when dealing with expected values for a risk factor whose distribution is not perfectly known. Given a random variable Y on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in a separable Hilbert space H endowed with its Borel σ-algebra $\mathcal{B}(H)$, one is usually interested in expressions of the form

$$\mathbb{E}_{\mathbb{P}}\big[f(Y)\big] = \int_H f(y)\,\mu(\mathrm{d}y),$$

where $f\colon H \to \mathbb{R}$ is a, say, continuous loss or payoff function and $\mu = \mathbb{P} \circ Y^{-1}$ is the distribution of Y, i.e., a probability measure on $\mathcal{B}(H)$.

However, in many situations, precise knowledge of the underlying distribution μ of Y may not be at hand, and only a rough estimate or an expert opinion suggesting a particular form of reference distribution μ may be available. This is a special instance of *model uncertainty* appearing, for example, in the context of catastrophic risk in reinsurance or default risk within large credit portfolios in banking. A standard way to deal with this type of uncertainty is to look at worst case losses among a set of plausible probability distributions. In our study, we follow this approach, and estimate worst case losses over balls around the reference model μ in the p-Wasserstein distance. This leads to an expression of the form

$$R(h)f := \sup_{\mathcal{W}_p(\mu,\nu)\le h} \int_H f(z)\,\nu(\mathrm{d}z), \tag{1}$$

where $h > 0$ can be interpreted as a level of uncertainty.

Functionals of the form (1) are widely studied in the context of distributionally robust optimization problems, see, for example, [1, 2, 3]. A standard approach to tackle the infinite-dimensional optimization related to (1) is to look for a suitable dual formulation, for example, by transforming the primal problem into a superhedging problem. We look at this problem from a different angle. The key idea of our approach is to look for a set of parameters Θ and a parametric version $R_\Theta(h)$ of the functional (1), which provides a first order approximation of $R(h)$ as the level of uncertainty h tends to zero.

As an extension, we introduce an additional mean constraint in the optimization (1). This constraint enters naturally when dealing with risk-neutral pricing, where the mean of the underlying is assumed to be known (e.g., by standard non-arbitrage arguments). Finally, we also impose a so-called martingale constraint on the functional $R(h)$. That is, we restrict the optimization to a set of probability measures, which are given in terms of a martingale perturbation of the reference measure or, in different words, measures that admit a martingale coupling with μ. This setup is closely intertwined with the topics of martingale optimal transport (MOT) and model-free pricing in mathematical finance. We provide an asymptotic parametrization of the constrained version of (1) through a suitable randomization of the reference measure.

References

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