# Expected Hitting Times for Continuous-Time Imprecise Markov Chains*

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**Preliminaries** Dynamical systems whose time domain is the non-negative reals $\mathbb{R}_{\geq 0}$ are often modelled using continuous-time Markov chains (CTMCs). Such a system is assumed to take values in a non-empty, finite state space $\mathcal{X}$ of size $n := |\mathcal{X}|$. When a CTMC $P$ is taken to be time-homogeneous, it is well-known that it can be described with a single $n \times n$ *rate matrix* $Q$, up to its initial distribution. Such a matrix satisfies $Q(x, y) \geq 0$ for all $x, y \in \mathcal{X}$ such that $x \neq y$, and $\sum_{y \in \mathcal{X}} Q(x, y) = 0$ for all $x \in \mathcal{X}$. An important inference for a CTMC are its *expected hitting times* $\mathbb{E}_P[\tau^A_{\mathbb{R}_{\geq 0}} | X_0]$ with respect to some $A \subset \mathcal{X}$, with $A \neq \emptyset$. These describe, in expectation, how long it will take for $P$ to reach a state in $A$ given that $P$ starts in $X_0$.

When confronted with uncertainty concerning the specification of a CTMC, one may choose to use a continuous-time *imprecise* Markov chain (CTIMC) instead. This uncertainty may be with respect to the numerical parameters (i.e., the entries of $Q$), or with respect to structural assumptions like Markovianity or time-homogeneity. We obtain a CTIMC by finding a *set* of stochastic processes 'consistent' with a set of rate matrices $\mathcal{Q}$ satisfying some regularity conditions (Krak [1]). We define three kinds of CTIMCs with respect to a given $\mathcal{Q}$, which each reflect different structural assumptions on the true process. These are the set $\mathcal{P}^{\mathrm{HM}}_{\mathcal{Q}}$ of processes which are time-homogeneous and Markovian, the set $\mathcal{P}^{\mathrm{M}}_{\mathcal{Q}}$ of processes which are Markovian but not necessarily time-homogeneous, and the set $\mathcal{P}^{\mathrm{I}}_{\mathcal{Q}}$ of processes which are not necessarily time-homogeneous or Markovian. For a CTIMC described by a set of processes $\mathcal{P}$, we can compute conservative lower and upper bounds that bound any desired inference, for each model $P \in \mathcal{P}$. For expected hitting times, these bounds are given by the *lower* (resp. *upper*) *expected hitting times* $\underline{h}(x) := \inf_{P \in \mathcal{P}} \mathbb{E}_P[\tau^A_{\mathbb{R}_{\geq 0}} | X_0 = x]$ (resp. $\overline{h}(x) := \sup_{P \in \mathcal{P}} \mathbb{E}_P[\tau^A_{\mathbb{R}_{\geq 0}} | X_0 = x]$), for all $x \in \mathcal{X}$.

**Previous Work** As is shown by Krak [1], both the lower and upper expected hitting times coincide for all three types of CTIMC. In addition to this, it is shown that the vectors $\underline{h}$ and $\overline{h}$ are the minimal non-negative solutions to two non-linear systems of equations. These results are obtained under the assumption that the set $A$ is so-called *lower reachable* from every state in $A^c := \mathcal{X} \setminus A$ [1, Assumption 2]. This condition essentially guarantees that the probability of eventually hitting $A$ is bounded away from zero, meaning that the expected hitting times remain bounded for all $P \in \mathcal{P}^{\mathrm{I}}_{\mathcal{Q}}$.

**Our Contribution** The aim of the present work is to remove this reachability condition and thereby generalise the results of Krak [1]. This is thought to be possible, as this has already been achieved in the discrete-time case [2]. So far, we have successfully removed the assumption for the results pertaining to the upper expected hitting times, characterising those states $x \in \mathcal{X}$ with $\overline{h}(x) = +\infty$, and obtaining a system of equations akin to the one found in the earlier work of Krak for the upper expected hitting times of the remaining states. Hence, for the upper expected hitting times, the results are complete! For the lower expected hitting times, the analysis is on-going. So far, we have focused on the subset $\mathcal{Z} \subseteq \mathcal{X}$ for which $A$ is lower reachable from every state, and from where no state from which $A$ is not lower reachable can be reached. For states in $\mathcal{Z}$, it has been shown that the lower expected hitting times are the same for all three types of Markov chains. In addition to this, a non-linear system similar to the system presented by Krak [1] has also been identified. It remains to expand the analysis of the lower expected hitting times to the states in $\mathcal{X} \setminus \mathcal{Z}$.

## References

[1] Thomas Krak. Hitting times for continuous-time imprecise-Markov chains. In *Proceedings of the Thirty-Eighth Conference on Uncertainty in Artificial Intelligence*, volume 180 of *PMLR*, pages 1031–1040. PMLR, 01–05 Aug 2022. URL https://proceedings.mlr.press/v180/krak22a.html.

[2] Thomas Krak, Natan T'Joens, and Jasper De Bock. Hitting times and probabilities for imprecise Markov chains. In *Proceedings of the Eleventh ISIPTA*, volume 103 of *PMLR*, pages 265–275. PMLR, 03–06 Jul 2019. URL https://proceedings.mlr.press/v103/krak19a.html.

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