A Constructive Theory for Conditional Lower Previsions Only Using Rational Valued Probability Mass Functions with Finite Support

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The usual interpretation of lower previsions goes via betting [5]. However, in high risk situations where the assessors themselves are at risk, some have argued that the betting interpretation of probability, and therefore also of lower previsions, cannot be applied [1]. For this reason, Lindley [3] stated an interpretation of probability based on urns, leading to a theory of rational-valued probabilities with finite support. Let Ω denote a (potentially infinite) sample space. Think of each $\omega \in \Omega$ as a colour. A subject can then specify a finite collection of coloured balls with colours from Ω, called an *urn*, as a standard for their uncertainty about $\omega \in \Omega$: they declare that their uncertainty about the colour of one ball uniformly drawn from their urn is equivalent to their uncertainty about the state of the world. Since one ball is drawn uniformly, only proportions matter, so it suffices for the subject to specify a rational-valued probability mass function on Ω with finite support.

In this work, we generalize Lindley's interpretation to arbitrary real-valued conditional lower previsions. This provides an alternative interpretation of lower previsions. Specifically, a subject's conditional lower prevision $\underline{P}$ is interpreted as them specifying an approximate lower arithmetic average for each (f, A) in the domain of $\underline{P}$ (where f is a gamble and A is an event), in the following sense:

> For every $\epsilon > 0$ and every finite subset $\mathcal{K}$ of the domain of $\underline{P}$, the subject's uncertainty is equivalent to drawing from an urn whose composition is such that, for every $(f, A) \in \mathcal{K}$, there is at least one ball with colour in A and the arithmetic average of the balls with colour in A, when labelled with f, is at least $\underline{P}(f \mid A) - \epsilon$.

We emphasize that, in this generalized interpretation, urns are still assumed to contain only finitely many balls. Therefore, only finitely many colours $\omega \in A$ can participate in the average (and at least one must participate), so the average is always defined. Interestingly, this interpretation can be shown to completely recover Williams's theory: it leads to new expressions for avoiding sure loss and natural extension which are mathematically equal to Williams's betting based equivalents.

A complication in the above interpretation is that the urn may depend on the choice of ϵ and $\mathcal{K}$. However, using the idealization axiom from non-standard analysis [4], the above interpretation can be rewritten to avoid this complication:

> The subject's uncertainty is equivalent to drawing from an urn whose composition is such that, for every standard $(f, A) \in \mathcal{K}$, there is at least one ball with colour in A and the arithmetic average of the balls with colour in A, when labelled with f, is at least $\underline{P}(f \mid A)$ up to an infinitesimal.

As an interesting corollary, we recover the well known result that every full conditional finitely additive probability measure can be represented by a single (typically, non-standard) probability mass function with finite support [2].

One key outcome is that the Williams natural extension for conditional lower previsions can be written as a limit of lower envelopes of sets of probability mass functions with finite support. This lower envelope theorem is fully constructive, in that it does not rely on the ultra-filter principle. It also provides a sensitivity analysis interpretation for conditional lower previsions that does not need full conditional finitely additive probability measures.

References

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