Testing the Coherence of Data and External Intervals via an Imprecise Sargan-Hansen Test

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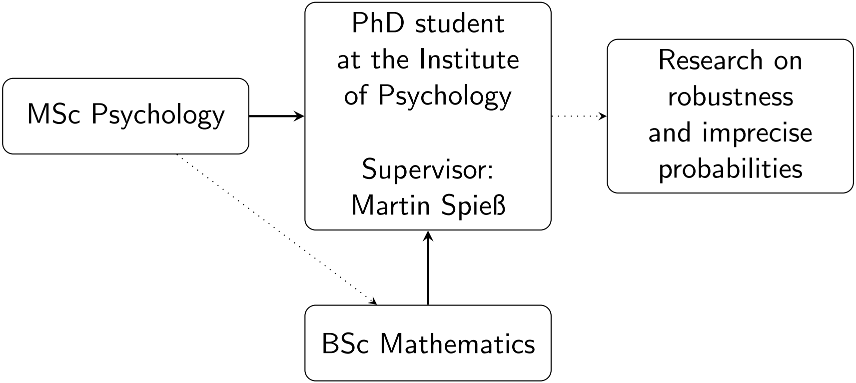
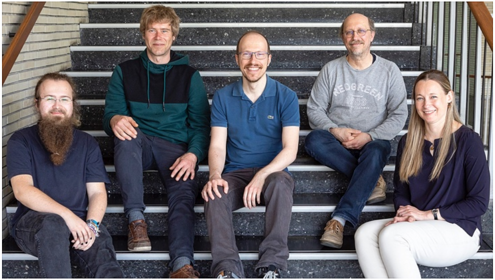
My Background



My Research Group

Departement of Research Methods and Statistics, Institute of Psychology, Universität Hamburg
Head: Martin Spieß (top right)



Research Interests of my Research Group

- Semiparametric estimation of panel or repeated measures models

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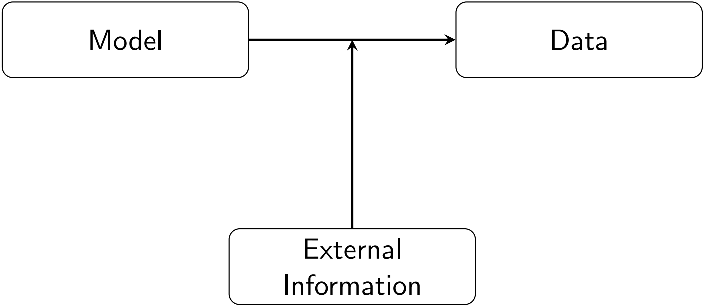
- **Robust use of external information in (frequentist) statistical inference**

External Information in Statistical Inference



Generalized Method of Moments (GMM)

- Developed by Hansen (1982) to estimate model parameter $\boldsymbol{\theta}$ based on a sample $\mathbf{z}_1, \ldots, \mathbf{z}_n$ through a potentially overidentified system of (sample) moment conditions $\frac{1}{n}\sum_{i=1}^{n} \mathbf{m}(\mathbf{z}_i, \boldsymbol{\theta}) = 0$

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- Example: $\frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i(y_i - \mathbf{x}_i\boldsymbol{\theta}) = 0$ for simple linear regression
- The estimation is done by minimizing a quadratic form $\hat{Q}_n(\boldsymbol{\theta}) = (\frac{1}{n}\sum_{i=1}^{n} \mathbf{m}(\mathbf{z}_i, \boldsymbol{\theta}))^T \hat{\mathbf{W}}(\frac{1}{n}\sum_{i=1}^{n} \mathbf{m}(\mathbf{z}_i, \boldsymbol{\theta}))$, where $\hat{\mathbf{W}}$ is a weighting matrix

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- Scope: OLS, maximum likelihood estimation (via score function), M-estimation, Generalized Estimating Equations (GEE)

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- Example 2: $\frac{1}{n}\sum_{i=1}^{n} sign(y_i - 100) = 0$ for $Median(Y) = 100$
- **Main Idea (Part 1): Use the GMM framework to incorporate external information**

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- **Main Idea (Part 2): Use intervals instead of points for the external information**
  $\Rightarrow$ This naturally leads to an imprecise probability situation, via coarse data completion (Augustin et al., 2014)

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- The GMM framework includes a test that can serve this purpose: the Sargan-Hansen test (based on $n\hat{Q}_n(\hat{\theta}) \xrightarrow{d} \chi^2_r$)
- The Sargan-Hansen test is a test for overidentifying moment conditions that shows "how far" the system is away from being identified

The Sargan-Hansen Test Based on external intervals

Let $\mathbf{e}_0$ be the true value of the external moments and $\mathbf{I}_{ex}$ be the external interval.

Desired property of a coherence test:
Test decides that $\mathbf{e}_0 \notin \mathbf{I}_{ex} \Rightarrow$ Test decides that $\mathbf{e}_0 \notin \mathbf{I}$ for all $\mathbf{I} \subset \mathbf{I}_{ex}$

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Results:

- For many relevant cases, the model moment conditions can be shortened from the formula for the test statistic (model independence)
- The imprecise Sargan-Hansen test based on a $\Gamma-$Maximin decision rule for selecting a p-value has the desired property
- In most cases, the simulations performed showed good behavior in terms of type I errors and power

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- maximal lower probality is reached at the event $\{Q > \underline{Q}\}$ (monotonicity)
  $\Rightarrow P_{\chi^2_{p_2}}(\{Q > \underline{Q}\})$ **is the p-value for the imprecise Sargan-Hansen test**

Outlook

The test construction is mainly based on the stochastic ordering of the distribution family underlying the test statistic.

⇒ The arguments can be generalized to many statistical tests used in psychology and econometrics, e.g., the Wald test for general linear and nonlinear hypotheses, the likelihood ratio test as well as the Langrange multiplier test.

Thank you for your attention!

If you want to see and discuss the (mathematical) details, feel free to do so at my poster presentation!

References

Augustin, T., Walter, G., & Coolen, F. P. A. (2014). Statistical inference. *Introduction to imprecise probabilities* (pp. 135–189). John Wiley & Sons, Ltd. https://doi.org/10.1002/9781118763117.ch7

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