Evaluating Imprecise Forecasts

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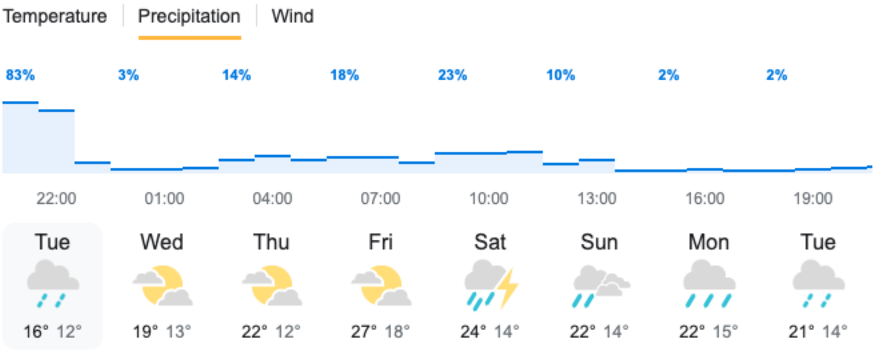
Motivation

Probabilistic Forecasting



Penalise “inaccurate forecasts” using **loss functions**

- **Ex Ante: incentivise “careful” and “honest” forecasts**
- **Ex Post: sift accurate/inaccurate forecasters; how to improve**

Imprecise Forecasting

			
X	3	2	-1
Y	-50	-10	10

Is X desirable? Is Y desirable?

Ex Ante **Ex Post**

Ex Ante: Prediction Markets

GOOD JUDGMENT

Will Russia and Ukraine sign or announce an agreement to end the current conflict in Ukraine?

A	Before 1 June 2022	1%	↓ 1
B	Between 1 June 2022 and 31 July 2022	4%	↓ 1
C	Between 1 August 2022 and 30 September 2022	7%	–
D	Between 1 October 2022 and 30 November 2022	9%	–
E	Not before 1 December 2022	79%	↑ 2

Probabilities as of 11 May 2022

When will Russia and Ukraine sign or announce an agreement to end the current conflict in Ukraine?

Opened: 14 Oct 2022, Suspends 30 Sep 2024

		Current Forecast	1-Week Change
A	Before 1 April 2023	1%	↓ 1
B	Between 1 April 2023 and 30 September 2023	7%	—
C	Between 1 October 2023 and 31 March 2024	13%	↓ 1
D	Between 1 April 2024 and 30 September 2024	18%	—
E	Not before 1 October 2024	61%	↑ 2

Probabilities as of 6 January 2023

Implied Median: 18 Dec 2024

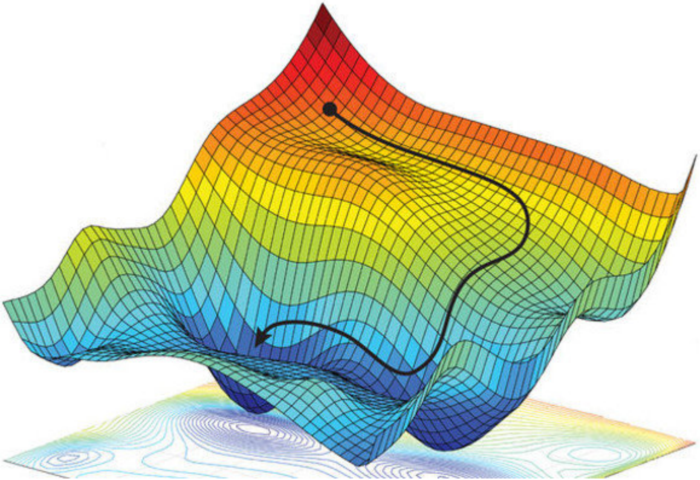
Source: Good Judgment Inc

- Traders can change market IP forecasts; pay score of status quo forecast in exchange for score of updated forecast
- Incentivise reporting of “best” IP model from class of admissible models

Ex Post: Machine Learning

Label	Description
0	T-shirt/top
1	Trouser
2	Pullover
3	Dress
4	Coat
5	Sandal
6	Shirt
7	Sneaker
8	Bag
9	Ankle boot

Ex Post: Machine Learning



IP Loss Functions

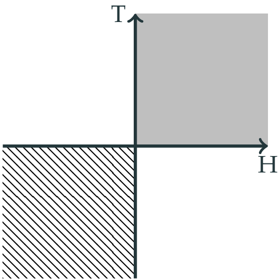
Let Ω be a finite possibility space.

A gamble $X : \Omega \to \mathbb{R}$ is an uncertain reward. We will treat them as elements $X = \langle x_1, \ldots, x_n \rangle$ of $\mathbb{R}^n$.

A set $\mathcal{D} \subseteq \mathbb{R}^n$ is a **coherent set of almost desirable gambles** if and only if it satisfies the following five axioms:

AD1. If $X < 0$ then $X \notin \mathcal{D}$ (where $X < 0 \Leftrightarrow x_i < 0$ for all $i \leq n$)

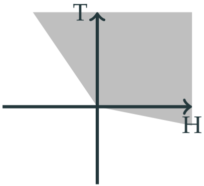
AD2. If $X \geq 0$ then $X \in \mathcal{D}$ (where $X \geq 0 \Leftrightarrow x_i \geq 0$ for all $i \leq n$)



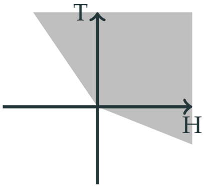
AD3. If $X \in \mathcal{D}$ and $\lambda > 0$ then $\lambda X \in \mathcal{D}$

AD4. If $X, Y \in \mathcal{D}$ then $X + Y \in \mathcal{D}$

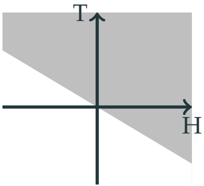
AD5. If $X + \epsilon \in \mathcal{D}$ for all $\epsilon > 0$ then $X \in \mathcal{D}$



convex cone including $\mathbb{R}^2_{\geq 0}$
excluding $\mathbb{R}^2_{<0}$



larger set:
more committal model



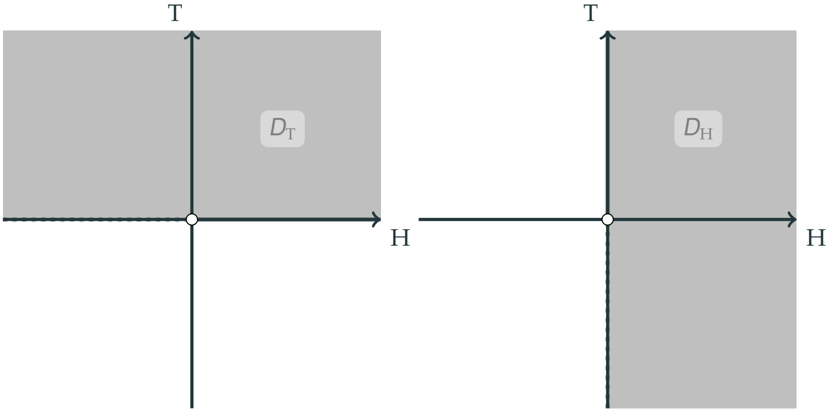
open half-space:
precise prob model

Ideal Sets of Gambles

The ideal set of almost desirable gambles if ω_i is true is given by

$$\mathcal{D}_i = \{X \mid x_i \geq 0\}$$

$\mathcal{D}_i$ contains all and only the gambles that are *in fact* almost desirable at ω_i.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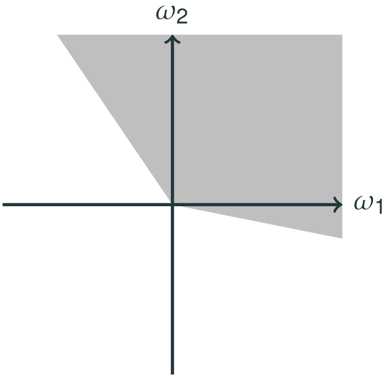
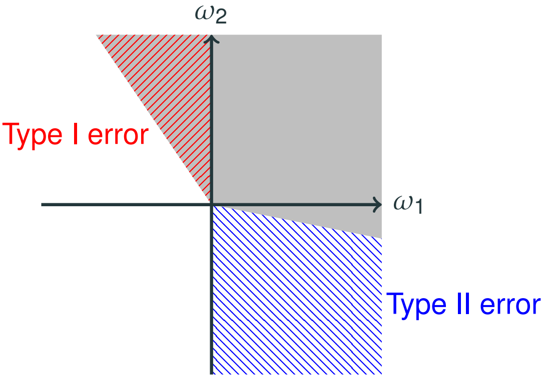
Type 1 and 2 Error





Type I and type II error for world ω_1

$\mathcal{E}_i = T1 \bigcup T2$ is $\mathcal{D}$'s **error set** at ω_i

- The set of gambles that $\mathcal{D}$ mischaracterizes at ω_i.

Inaccuracy is a measure of error.

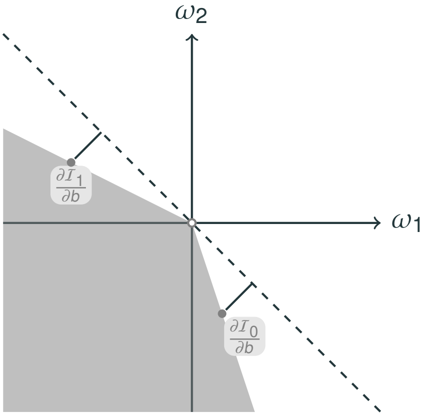
The inaccuracy of $\mathcal{D}$ at ω_i, $\mathcal{I}(\mathcal{D}, \omega_i)$, is the measure of $\mathcal{E}_i$ according to an appropriate measure ν_i:

$$\mathcal{I}(\mathcal{D}, \omega_i) = \mathcal{I}_i(\mathcal{D}) = \nu_i(\mathcal{E}_i)$$

Assume that ν_i is finite and absolutely continuous with respect to the product Lebesgue measure μ. In that case

$$\mathcal{I}_i(\mathcal{D}) = \int_{\mathcal{E}_i} |\phi_i| \, \mathrm{d}\mu$$

Main Result



If ν_i is finite and absolutely continuous with respect to μ, for all $i \leq n$, then the following two conditions are equivalent:

1. There is some $h : \mathbb{R}^{n-1} \to \mathbb{R}$ s.t. for all $i \leq n$

$$\delta \mathcal{I}_i(b, h) = \int_{\mathbb{R}^{n-1}} \frac{\partial \mathcal{I}_i}{\partial b} h \, \mathrm{d}\lambda < 0$$

   First variation—calculus of variations analogue of directional derivative

2. $0 \notin \operatorname{posi}\left(\left\{\phi_i(\cdot, b(\cdot)) \mid i \leq n\right\}\right)$

Loss functions for (infinitely many) precise previsions

Suppose that for all $i \leq n$

$$\mathcal{I}_i(\mathcal{D}) = \int_{\mathcal{E}_i} |g_i| \, \mathrm{d}\mu$$

Then for any probability mass function $p : \Omega \to \mathbb{R}$ and any $\mathcal{D} \neq \mathcal{D}_p$

$$\sum_{i \leq n} p_i \mathcal{I}_i(\mathcal{D}_p) < \sum_{i \leq n} p_i \mathcal{I}_i(\mathcal{D})$$

unless both $\mathcal{D} \setminus \mathcal{D}_p$ and $\mathcal{D}_p \setminus \mathcal{D}$ are sets of measure zero.

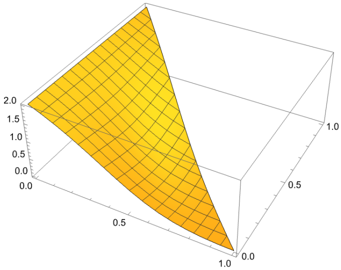
$\mathcal{I}$ is a **strictly proper scoring rule**.

Loss functions for (infinitely many) precise previsions

Example. Let ρ be the normal distribution on the Borel σ-algebra $\mathfrak{B}(\mathbb{R})$ with mean 0 and standard deviation 5. Let μ be the product measure $\rho \times \rho \times \rho$ on $\mathfrak{B}(\mathbb{R}^3)$. In that case

$$\mathcal{I}_i(\mathcal{D}_p) = \frac{5\left(1 - \frac{p_i}{\sqrt{p_1^2+p_2^2+p_3^2}}\right)}{2\pi}$$

This is a non-additive analogue of the Spherical score.



Loss functions for non-maximal sets of almost desirable gambles

Suppose that there are $\lambda \geq \gamma > 0$ such that

$$\mathcal{I}_i(\mathcal{D}) = \int_{\mathcal{E}_i} \begin{cases} -\lambda x_i & \text{if } x_i < 0 \\ \gamma x_i & \text{if } x_i \geq 0 \end{cases} \mathrm{d}\mu$$

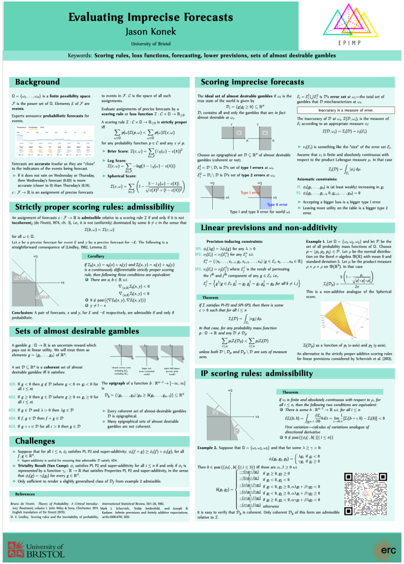
for all $i \leq n$. Then $\mathcal{D}_b = \{\langle g_1, g_2, g_3 \rangle \,|\, g_3 \geq b(g_1, g_2)\} \subseteq \mathbb{R}^3$ is admissible, *i.e.*, not dominated by some $\mathcal{D} \neq \mathcal{D}_b$, iff there are $\alpha, \beta \geq 0$ s.t.

$$b(x_1, x_2) = \begin{cases} \frac{-\gamma(\alpha x_1 + \beta x_2)}{\lambda} & \text{if } x_1 \geq 0, x_2 \geq 0 \\ \frac{-\lambda(\alpha x_1 + \beta x_2)}{\gamma} & \text{if } x_1 < 0, x_2 < 0 \\ \frac{-(\alpha\lambda x_1 + \beta\gamma x_2)}{\gamma} & \text{if } x_1 < 0, x_2 \geq 0, \alpha\lambda x_1 + \beta\gamma x_2 < 0 \\ \frac{-(\alpha\lambda x_1 + \beta\gamma x_2)}{\lambda} & \text{if } x_1 < 0, x_2 \geq 0, \alpha\lambda x_1 + \beta\gamma x_2 \geq 0 \\ \frac{-(\alpha\gamma x_1 + \beta\lambda x_2)}{\gamma} & \text{if } x_1 \geq 0, x_2 < 0, \alpha\gamma x_1 + \beta\lambda x_2 < 0 \\ \frac{-(\alpha\gamma x_1 + \beta\lambda x_2)}{\lambda} & \text{otherwise} \end{cases}$$

Loss functions for non-maximal sets of almost desirable gambles

EPIMP

Come to the poster!



References