A Nonstandard Approach To Stochastic Processes Under Probability Bounding

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Internal set theory

What is internal set theory?

- new predicate '**standard**' that applies to objects (sets, functions, . . .)
  - internal formulas: do not use 'standard' "$0 + 1$ equals 1"
  - external formulas: do use 'standard' "0 is a standard natural number"
- three new axioms added to ZFC to govern use of this predicate [Nel87]
  1. idealisation
  2. standardisation
  3. transfer

Internal set theory

Intuition?

- for an object to be standard, intuitively, we mean [DD95, §1.1.1, p. 2]:

  ‘at any stage within the mathematical discourse, [...]
  uniquely defined [using an explicitly written internal formula]’

- the new axioms capture this intuition, e.g.
  - if an object is uniquely defined by an internal formula, then it is standard
  - if an internal statement holds for all standard objects, then it holds for all objects
  - there are only finitely many standard objects

    more precisely, every set has a finite subset that contains all of its standard elements

Internal set theory

Why is it useful?

- has the notion of an **infinitesimal** (goes back to Newton, Leibniz, Cauchy, etc.; Weierstrass and others sadly failed to formalize these [BK12])
- many objects have an infinitely close standard object, called its **shadow**: allows us to move between standard functions with infinite domain and non-standard functions with finite domain

  very useful to study stochastic processes in continuous time!

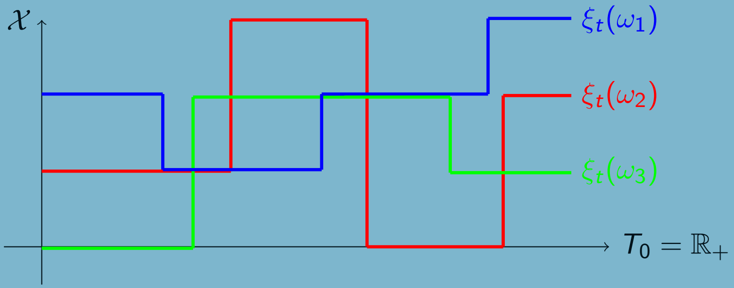
Difficulties

- for historical reasons, most mathematicians are unfamiliar with it
- inherent ambiguous boundary between standard and non-standard objects

Stochastic processes

What is a stochastic process?

- standard finite state space $\mathcal{X}$, standard index set $T_0 = \mathbb{N}$ or $T_0 = \mathbb{R}_+$
- standard possibility space Ω
- standard function $\xi\colon T_0 \times \Omega \to \mathcal{X}$



Stochastic processes

What is a stochastic process?

For any $T \subseteq T_0$, we define

- $\mathcal{A}(T)$: algebra of events generated by $\{\omega \colon \xi_t(\omega) = x\}$ for $t \in T$
- $\mathcal{L}(T)$: gambles formed by linear span of $\mathcal{A}(T)$
- $\mathcal{K}(T) \coloneqq \mathcal{L}(T) \times (\mathcal{A}(T) \setminus \{\emptyset\})$

Definition

A *process* on T is a coherent lower prevision $\underline{\mathbb{E}}$ defined on $\mathcal{K}(T)$.

Stochastic processes

Key Assumption

Let T be a **finite** subset of T_0 **containing all standard elements** of T_0.

Shadow of a process

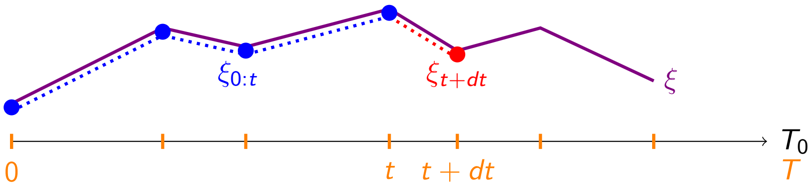
Theorem

For every process $\mathbb{E}$ on T, there is a unique standard process $\mathbb{E}_0$ on T_0, called **shadow** *of $\mathbb{E}$, satisfying*

$$\mathbb{E}(f \mid A) \simeq \mathbb{E}_0(f \mid A) \text{ for all standard } (f, A) \in \mathcal{K}(T_0) \tag{1}$$

Vice versa, every standard process $\mathbb{E}_0$ on T_0 is the shadow of some elementary process $\mathbb{E}$ on T.

Imprecise Markov chains



Imprecise Markov chains: Notation

- if $t \in T \setminus \{\max T\}$ then $t + dt$ denotes the successor of t in T, i.e.

$$dt := \min\{t' \in T : t' > t\} - t \tag{2}$$

- for any function ϕ on T, $\phi_{0:t}$ denotes the restriction of ϕ to $[0, t] \cap T$

Imprecise Markov chains

Imprecise Markov chains: Definition

Definition

A precise elementary process $\mathbb{E}$ on T is **compatible** with $(\underline{\mathbb{I}}, \underline{\mathbb{T}})$ if for all paths $x\colon T \to \mathcal{X}$, all $t \in T \setminus \{\max T\}$, and all $f \in \mathbb{R}^{\mathcal{X}}$,

$$\mathbb{E}(f(\xi_0) \geq \underline{\mathbb{I}}(f) \tag{3}$$

$$\mathbb{E}(f(\xi_{t+dt}) \mid \xi_{0:t} = x_{0:t}) \geq \underline{\mathbb{T}}_t(f)(x_t) \tag{4}$$

If $\underline{\mathbb{E}}$ denotes the lower envelope of all these compatible precise elementary processes, then $\underline{\mathbb{E}}$ is called the **elementary imprecise Markov chain** induced by $(\underline{\mathbb{I}}, \underline{\mathbb{T}})$. Its shadow is called the **imprecise Markov chain** induced by $(\underline{\mathbb{I}}, \underline{\mathbb{T}})$.

Continuous time imprecise Markov chains

Continuous time imprecise Markov chains

- fix a standard **lower rate operator** $\underline{\mathbb{Q}}\colon \mathbb{R}^{\mathcal{X}} \to \mathbb{R}^{\mathcal{X}}$ and define for $t \in T \setminus \{\max T\}$

$$\underline{\mathbb{T}}_t := I + dt\underline{\mathbb{Q}} \tag{5}$$

- the shadow $\underline{\mathbb{E}}_0$ of this elementary imprecise Markov chain $\underline{\mathbb{E}}$ behaves just like the usual continuous time imprecise Markov chains

Theorem

For all $x \in \mathcal{X}$, $f \in \mathbb{R}^{\mathcal{X}}$, and $t \geq 0$, $\underline{\mathbb{E}}_0(f(\xi_t) \mid \xi_0 = x) = e^{t\underline{\mathbb{Q}}}(f)(x)$.

Proof.

See poster. □

Conclusion

Conclusion

- simplified modelling of imprecise processes
- ‘well behaved’ assumption not needed [KBS17]
- so far only Williams coherence
- Nelson’s more advanced model?
- **time to embrace old school maths?**

Come discuss at the poster!

References I

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