The background is a dark brown color with various yellow and white mathematical scribbles. At the top left, there is a scribble that looks like $\xi - \eta(\dots) \rightarrow$. At the top right, there is a large scribble that looks like $\xi (-1 \cdot 0 - - \xi) \dots$. In the bottom right corner, there are several scribbles including a plus sign $+$, a less-than sign $<$, and some other symbols. The title text is centered in a light green rounded rectangle.

A Nonstandard Approach To Stochastic Processes Under Probability Bounding

Matthias C. M. Troffaes

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Intuition?

- ▶ for an object to be standard, intuitively, we mean [DD95, §1.1.1, p. 2]:
‘at any stage within the mathematical discourse, [...] uniquely defined [using an explicitly written internal formula]’
- ▶ the new axioms capture this intuition, e.g.
 - ▶ if an object is uniquely defined by an internal formula, then it is standard
 - ▶ if an internal statement holds for all standard objects, then it holds for all objects
 - ▶ there are only finitely many standard objectsmore precisely, every set has a finite subset that contains all of its standard elements

• 0 • 1 • 2 • 3 • ... • ... • ... • $n-1$ • n • $n+1$ • ...

Why is it useful?

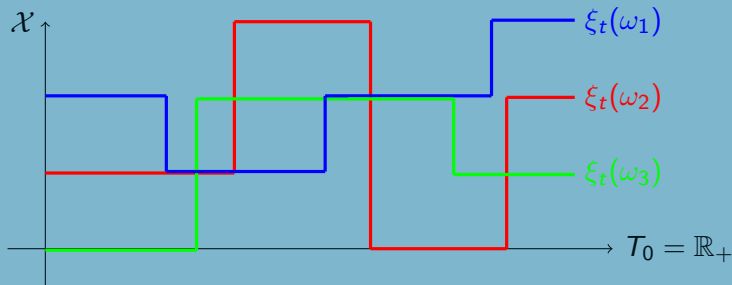
- ▶ has the notion of an **infinitesimal** (goes back to Newton, Leibniz, Cauchy, etc.; Weierstrass and others sadly failed to formalize these [BK12])
 - ▶ many objects have an infinitely close standard object, called its **shadow**: allows us to move between standard functions with infinite domain and non-standard functions with finite domain
- very useful to study stochastic processes in continuous time!

Difficulties

- ▶ for historical reasons, most mathematicians are unfamiliar with it
- ▶ inherent ambiguous boundary between standard and non-standard objects

What is a stochastic process?

- ▶ standard finite state space \mathcal{X} , standard index set $T_0 = \mathbb{N}$ or $T_0 = \mathbb{R}_+$
- ▶ standard possibility space Ω
- ▶ standard function $\xi: T_0 \times \Omega \rightarrow \mathcal{X}$



What is a stochastic process?

For any $T \subseteq T_0$, we define

- ▶ $\mathcal{A}(T)$: algebra of events generated by $\{\omega: \xi_t(\omega) = x\}$ for $t \in T$
- ▶ $\mathcal{L}(T)$: gambles formed by linear span of $\mathcal{A}(T)$
- ▶ $\mathcal{K}(T) := \mathcal{L}(T) \times (\mathcal{A}(T) \setminus \{\emptyset\})$

Definition

A *process* on T is a coherent lower prevision $\underline{\mathbb{E}}$ defined on $\mathcal{K}(T)$.

Stochastic processes

Key Assumption

Let T be a **finite** subset of T_0 **containing all standard elements** of T_0 .

Shadow of a process

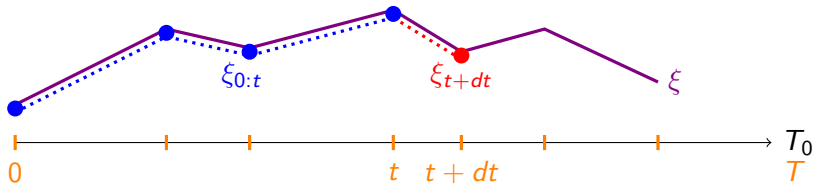
Theorem

For every process $\underline{\mathbb{E}}$ on T , there is a unique standard process $\underline{\mathbb{E}}_0$ on T_0 , called **shadow** of $\underline{\mathbb{E}}$, satisfying

$$\underline{\mathbb{E}}(f \mid A) \simeq \underline{\mathbb{E}}_0(f \mid A) \text{ for all standard } (f, A) \in \mathcal{K}(T_0) \quad (1)$$

Vice versa, every standard process $\underline{\mathbb{E}}_0$ on T_0 is the shadow of some elementary process $\underline{\mathbb{E}}$ on T .

Imprecise Markov chains



Imprecise Markov chains: Notation

- ▶ if $t \in T \setminus \{\max T\}$ then $t + dt$ denotes the successor of t in T , i.e.

$$dt := \min\{t' \in T : t' > t\} - t \quad (2)$$

- ▶ for any function ϕ on T , $\phi_{0:t}$ denotes the restriction of ϕ to $[0, t] \cap T$

Imprecise Markov chains: Definition

Definition

A precise elementary process \mathbb{E} on T is **compatible** with $(\underline{\mathbb{I}}, \underline{\mathbb{T}})$ if for all paths $x: T \rightarrow \mathcal{X}$, all $t \in T \setminus \{\max T\}$, and all $f \in \mathbb{R}^{\mathcal{X}}$,

$$\mathbb{E}(f(\xi_0)) \geq \underline{\mathbb{I}}(f) \quad (3)$$

$$\mathbb{E}(f(\xi_{t+dt}) \mid \xi_{0:t} = x_{0:t}) \geq \underline{\mathbb{T}}_t(f)(x_t) \quad (4)$$

If $\underline{\mathbb{E}}$ denotes the lower envelope of all these compatible precise elementary processes, then $\underline{\mathbb{E}}$ is called the **elementary imprecise Markov chain** induced by $(\underline{\mathbb{I}}, \underline{\mathbb{T}})$. Its shadow is called the **imprecise Markov chain** induced by $(\underline{\mathbb{I}}, \underline{\mathbb{T}})$.

Continuous time imprecise Markov chains

Continuous time imprecise Markov chains

- ▶ fix a standard **lower rate operator** $\underline{Q}: \mathbb{R}^{\mathcal{X}} \rightarrow \mathbb{R}^{\mathcal{X}}$ and define for $t \in T \setminus \{\max T\}$

$$\underline{T}_t := I + dt\underline{Q} \quad (5)$$

- ▶ the shadow $\underline{\mathbb{E}}_0$ of this elementary imprecise Markov chain $\underline{\mathbb{E}}$ behaves just like the usual continuous time imprecise Markov chains

Theorem

For all $x \in \mathcal{X}$, $f \in \mathbb{R}^{\mathcal{X}}$, and $t \geq 0$, $\underline{\mathbb{E}}_0(f(\xi_t) \mid \xi_0 = x) = e^{t\underline{Q}}(f)(x)$.

Proof.

See poster. □

Conclusion

Conclusion

- ▶ simplified modelling of imprecise processes
- ▶ 'well behaved' assumption not needed [KBS17]
- ▶ so far only Williams coherence
- ▶ Nelson's more advanced model?
- ▶ **time to embrace old school maths?**

Come discuss at the poster!

A Nonstandard Approach To Stochastic Processes Under Probability Bounding

Matthew C. Taiters Durham University, UK

What is Interval set theory?

- new predicate **'interval'** that applies to objects (sets, functions, ...)
- these new objects called **ITs** to govern use of the predicate

Intuition?

- Interval formulae do not use 'interval' (ITs) to model IT
- Interval formulae do use 'interval' (ITs) to model interval number
- to an object to be the object, interval, etc. make ITs, ITs, ITs
- of any object with the nonstandard formulae [1]
- uniquely defined (using an explicitly written interval formulae)

Why is it useful?

- that the notion of an **interval** in the theory formulae intuition that gives back to Nelson's Logic, Coarsely etc [2]
- many objects have an infinity close standard object, called its **shadow**, allows us to model between standard functions with infinite domain and non-standard functions with finite domain
- very useful to study stochastic processes in continuous time!

Difficulties

- for historical reasons, most mathematicians are unfamiliar with it
- exploration of new systems needs care, require some technical logic
- legal set formation: cannot form sets with interval formulae

What is a stochastic process?

Notation

- standard finite state space T
- index set $\mathbb{I}_T \subseteq \mathbb{R}$
- probability space Ω
- function $\mathbb{I}_T \times \Omega \rightarrow T$ (given outcome ω at time t , the state is $\Omega(t, \omega)$)
- $\mathbb{I}_T \subseteq \mathbb{R}$, any subset of \mathbb{I}_T , e.g. finite intervals
- $\mathbb{A}(T)$ algebra generated by events of the form $\{\Omega(t) = a\}$
- $\mathbb{C}(T)$ linear space spanned by $\mathbb{A}(T)$
- $\mathbb{N}(T) = \mathbb{C}(T) \cap \mathbb{A}(T)$ (IT)

Definition

A (stochastic) process on T is a coherent lower prevision \mathbb{P} defined on $\mathbb{C}(T)$.

Nearly elementary processes

Theorem

Assume $\mathbb{I}_T \subseteq \mathbb{R}$ and T contains all adjacent elements of \mathbb{I}_T . Let $\mathbb{I}_T \subseteq \mathbb{R}$ and T be any process. Then there is a unique standard process $\mathbb{P}_s(\cdot | \mathbb{I}_T) \rightarrow \mathbb{P}$, called the **shadow** of \mathbb{P} , satisfying

$$\forall t, A \in \mathbb{A}(T), B \in \mathbb{A}(T) \quad \mathbb{P}(A) \leq \mathbb{P}(B | A) \quad (1)$$

Definition

A standard process \mathbb{P}_s on T_s is said to be nearly an elementary process \mathbb{P} on $T \in T_s$, T_s contains all standard elements of T_s , and \mathbb{P}_s is the shadow of \mathbb{P} .

Imprecise Markov chains: Notation

- T is a finite subset of \mathbb{I}_T consisting of standard elements of \mathbb{I}_T
- $T' = T \cup \{t\}$
- $t \in T'$ then t is called the successor of T in T' , i.e.
- $\text{succ}(T) = T' \setminus T = \{t\}$
- for any function \mathbb{P} on $T \times \mathbb{C}(T)$ (IT) denotes the restriction of \mathbb{P} to $\mathbb{C}(T) \cap T$

Imprecise Markov chains: Definition

- \mathbb{P} : coherent lower prevision on \mathbb{I}^T (for each $t \in T'$ and $a \in T$)
- $\mathbb{P}(a | A)$, coherent lower prevision on \mathbb{I}^T (for each $t \in T'$ and $a \in T$)

Definition

A process elementary process \mathbb{P} on T is **elementary** with $\mathbb{P}(a | \text{succ}(T))$ for all paths $a \in T \times T'$, and all $a \in T'$.

$$\mathbb{P}(a | \text{succ}(T)) = \mathbb{P}(a | T) \quad (2)$$

$$\mathbb{P}(a | \text{succ}(T)) = \mathbb{P}(a | T) \quad (3)$$

If \mathbb{P} denotes the lower prevision of all those compatible process elementary previsions, then \mathbb{P} is called the **elementary imprecise Markov chain** induced by $\mathbb{P}(a | \text{succ}(T))$. Its shadow is called the **imprecise Markov chain** induced by $\mathbb{P}(a | \text{succ}(T))$.

Continuous time imprecise Markov chains

- to a standard lower prevision \mathbb{P} , $\mathbb{P}^t \rightarrow \mathbb{P}^t$ and define
- the shadow \mathbb{P}_s of this elementary imprecise Markov chain
- behaves just like the usual continuous time imprecise Markov chain

Theorem

Let $a \in T$, $t \in \mathbb{I}^T$, and $t' \in \mathbb{I}^T$ (IT) $(t' - t) \in \mathbb{I}^T$.

Sketch of proof

Assume \mathbb{P} is \mathbb{P} . By iteration, only need to establish the equality for standard t is sufficient to show that

$$\mathbb{P}(a | \text{succ}(T)) = \mathbb{P}(a | T) \quad (4)$$

Indeed, with $\mathbb{P} = \text{co-lim}_{t \rightarrow \infty} \mathbb{P}^t$,

$$\mathbb{P}(a | \text{succ}(T)) = \mathbb{P}(a | T) = \lim_{t \rightarrow \infty} \mathbb{P}^t(a | \text{succ}(T)) = \lim_{t \rightarrow \infty} \mathbb{P}^t(a | T) = \mathbb{P}(a | T) \quad (5)$$

$$\mathbb{P}(a | \text{succ}(T)) = \mathbb{P}(a | T) = \lim_{t \rightarrow \infty} \mathbb{P}^t(a | \text{succ}(T)) = \lim_{t \rightarrow \infty} \mathbb{P}^t(a | T) = \mathbb{P}(a | T) \quad (6)$$

since \mathbb{P} and \mathbb{P}^t are linked, and $\mathbb{P} = \mathbb{P}$ (compare with [2])

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