A Study of Jeffrey's Rule With Imprecise Probability Models

Enrique Miranda & Arthur Van Camp

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Oviedo & Bristol


Enrique Miranda


Ignacio Montes


Arthur Van Camp


Jason Konek


Kevin Blackwell


Giacomo Molinari


Zhenhua Li

Universidad de Oviedo

Department of Statistics and Operations Research

University of BRISTOL

EPIMP

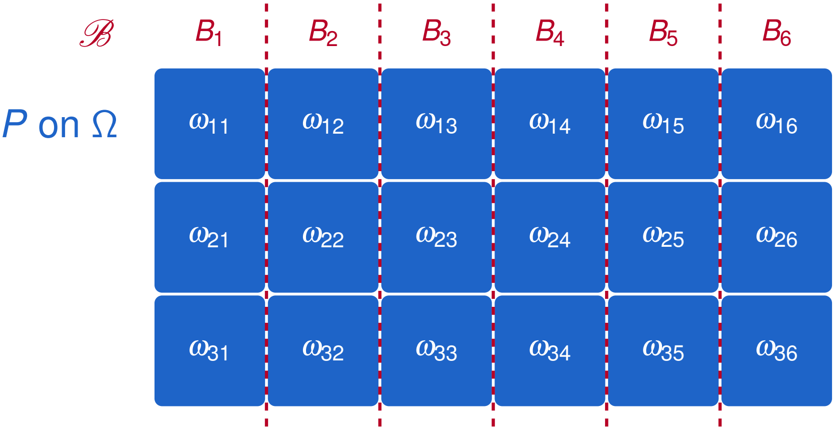
Epistemic Utility for Imprecise Probability

Bayes' Rule

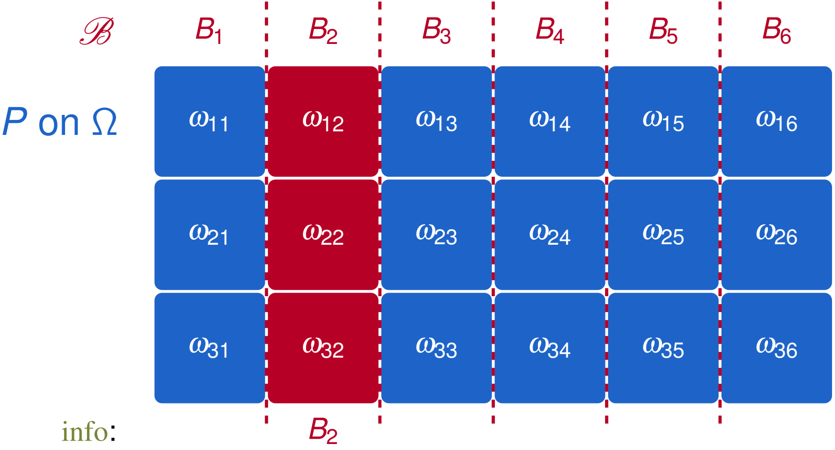
P on Ω

ω_{11}	ω_{12}	ω_{13}	ω_{14}	ω_{15}	ω_{16}
ω_{21}	ω_{22}	ω_{23}	ω_{24}	ω_{25}	ω_{26}
ω_{31}	ω_{32}	ω_{33}	ω_{34}	ω_{35}	ω_{36}

Bayes' Rule

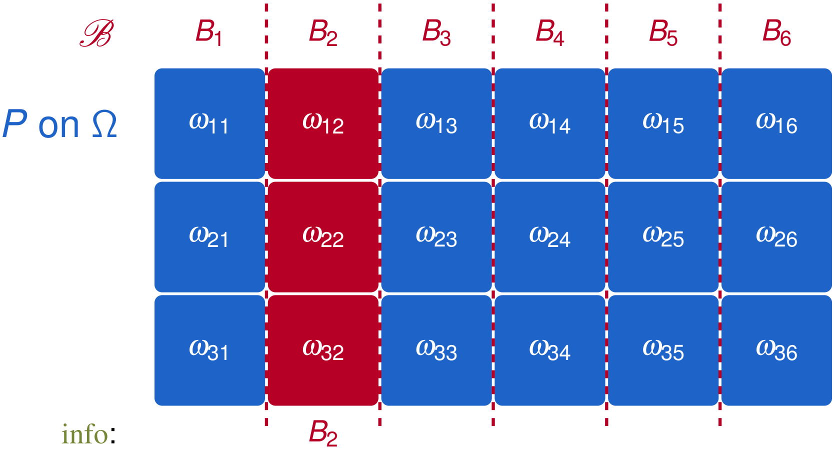

$\mathscr{B}$
B_1
B_2
B_3
B_4
B_5
B_6
P on Ω
ω_{11} ω_{12} ω_{13} ω_{14} ω_{15} ω_{16}
ω_{21} ω_{22} ω_{23} ω_{24} ω_{25} ω_{26}
ω_{31} ω_{32} ω_{33} ω_{34} ω_{35} ω_{36}

Bayes' Rule



$\Longrightarrow P_{\text{info}}$?

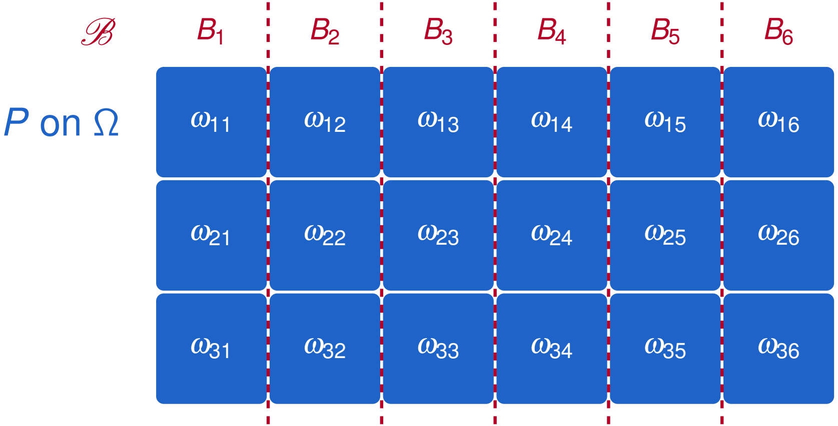
Bayes' Rule



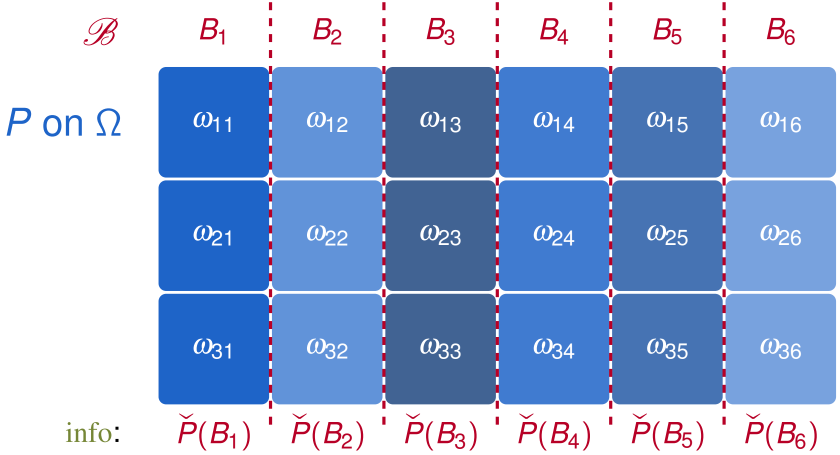
$$\Longrightarrow P_{\text{info}}?$$

$P_{\text{info}}(A) = P(A|B_2)$ using Bayes' Rule

Jeffrey's Rule

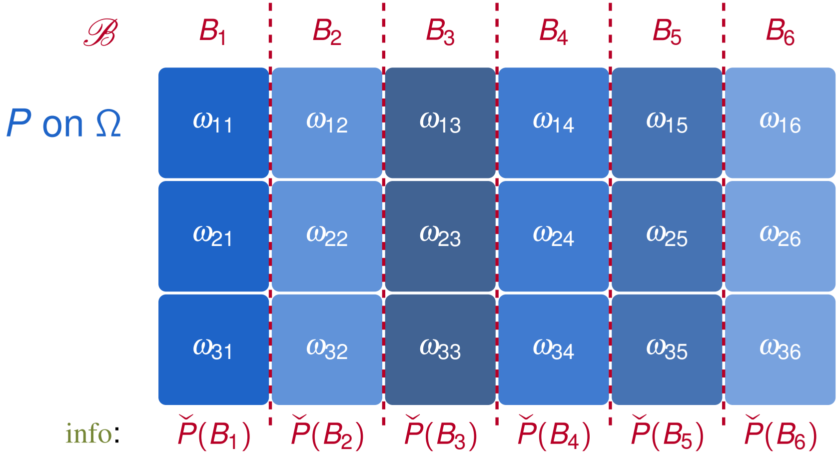


Jeffrey's Rule



$$\Longrightarrow P_{\text{info}}\,?$$

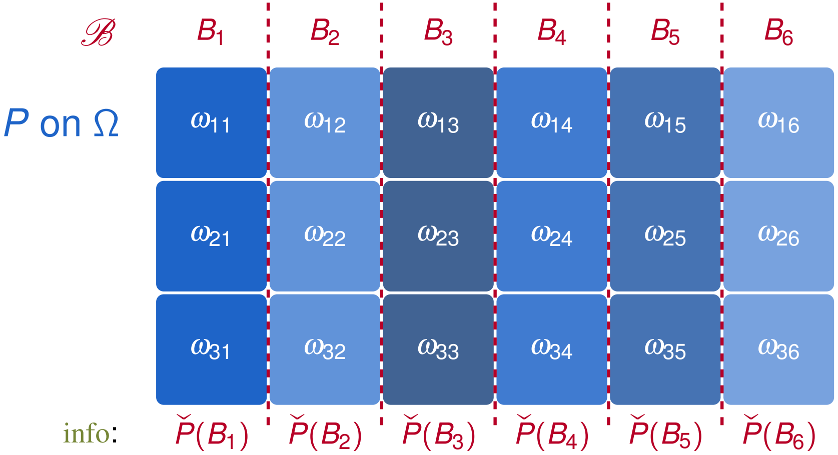
Jeffrey's Rule



$P_{\text{info}}(B) = \check{P}(B)$ for all $B \in \mathscr{B}$ [**agreeing on $\mathscr{B}$**]

$P_{\text{info}}(A|B) = P(A|B)$ for all $B \in \mathscr{B}, A \subseteq \Omega$ [**rigidity**]

Jeffrey's Rule



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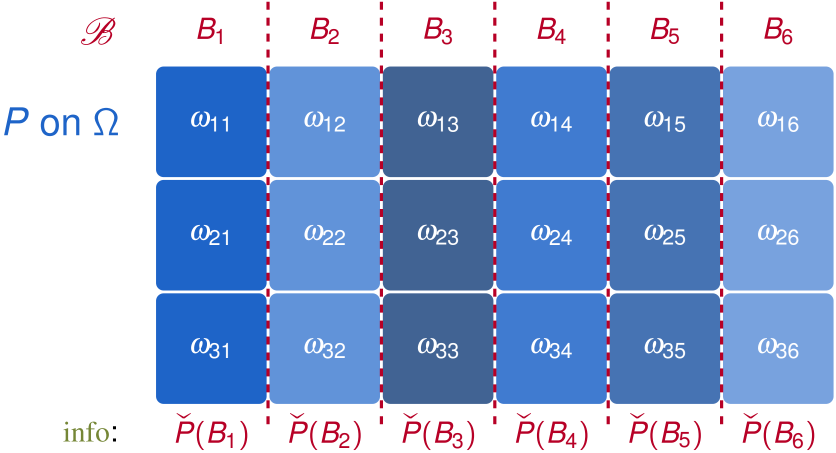
$\Rightarrow P_{\text{info}}(A) = \sum_{B \in \mathscr{B}} P_{\text{info}}(A|B) P_{\text{info}}(B)$

Jeffrey's Rule



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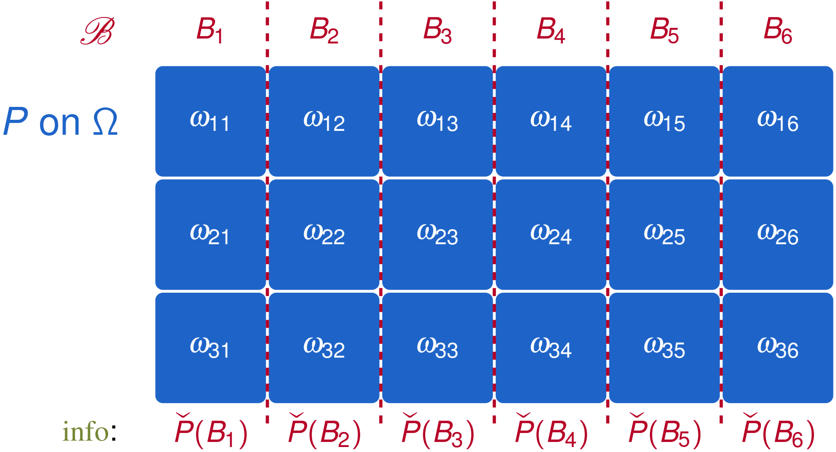
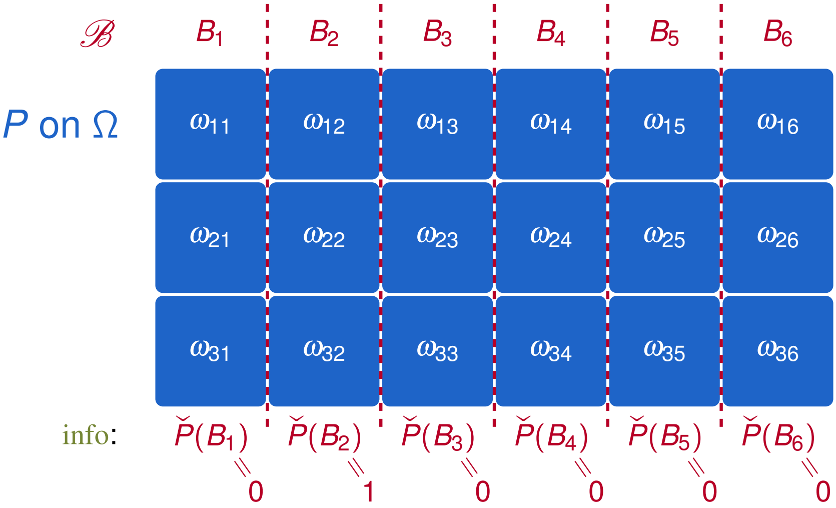
$\Rightarrow P_{\text{info}}(A) = \sum_{B \in \mathscr{B}} P(A|B)\check{P}(B)$

Bayes' Rule as Jeffrey's Rule



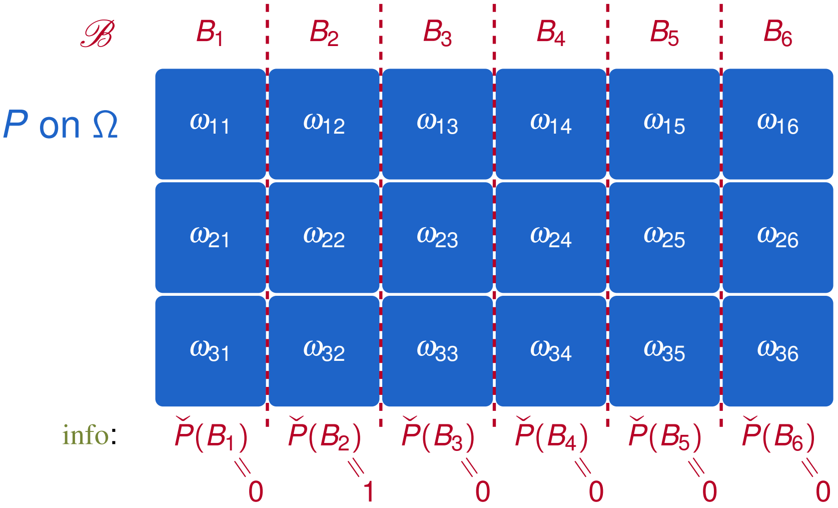
$$P_{\text{info}}(A) = \sum_{B \in \mathscr{B}} P(A|B)\check{P}(B)$$

Bayes' Rule as Jeffrey's Rule



$$P_{\text{info}}(A) = \sum_{B \in \mathscr{B}} P(A|B)\check{P}(B)$$

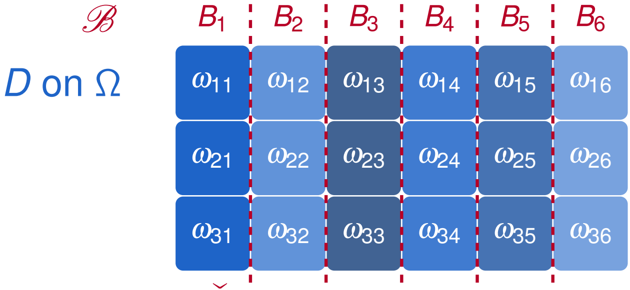
Bayes' Rule as Jeffrey's Rule



$$P_{\text{info}}(A) = \sum_{B \in \mathscr{B}} P(A|B)\check{P}(B) = P(A|B_1)0 + P(A|B_2)1 + P(A|B_3)0 + \cdots + P(A|B_6)0$$

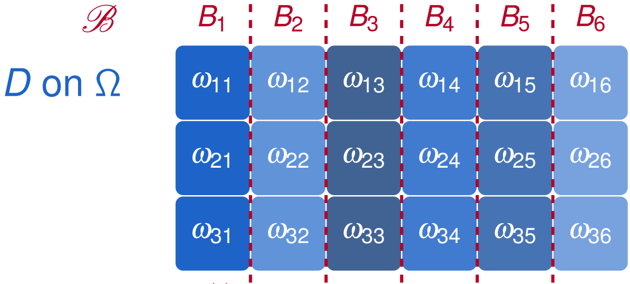
$$= P(A|B_2)$$

Jeffrey's Rule for desirability



info: $\check{D}$ consisting of gambles on $\mathscr{B}$

Jeffrey's Rule for desirability

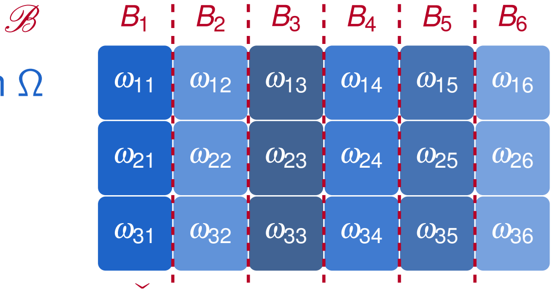


info: $\check{D}$ consisting of gambles on $\mathscr{B}$

$\Longrightarrow D_{\text{info}}$?

Jeffrey's Rule for desirability

D on Ω



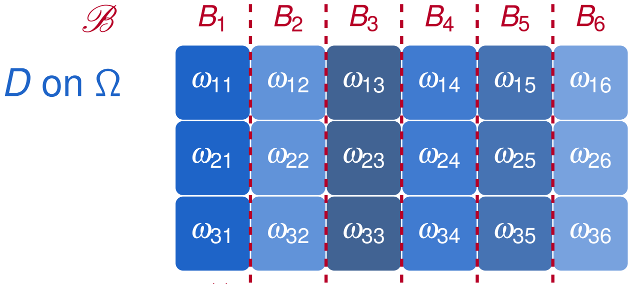
info: $\check{D}$ consisting of gambles on $\mathscr{B}$

Find the smallest coherent D_{info} such that

$D_{\text{info}} \supseteq \check{D}$ [**agreeing on $\mathscr{B}$**]

$D_{\text{info}}\rfloor B \supseteq D\rfloor B$ for all $B \in \mathscr{B}$ [**rigidity**]

Jeffrey's Rule for desirability



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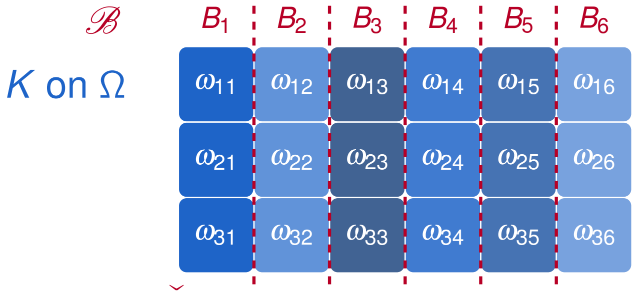
$\Rightarrow$ it follows from [*De Cooman & Hermans 2008*]

$$D_{\text{info}} = \operatorname{posi}\Big(\check{D} \cup \bigcup_{B \in \mathscr{B}} \mathbb{I}_B (D\rfloor B)\Big)$$

is the unique smallest coherent D_{info} that satisfies **agreeing on $\mathscr{B}$** and **rigidity**.

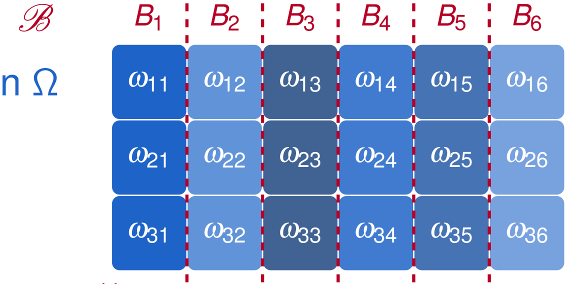
Jeffrey's Rule for choice functions



info: $\check{K}$ consisting of sets of gambles on $\mathscr{B}$

Jeffrey's Rule for choice functions

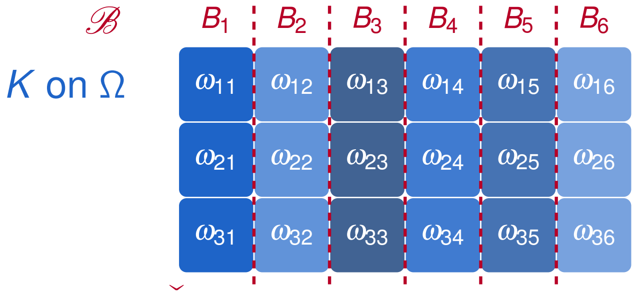
K on Ω



info: $\check{K}$ consisting of sets of gambles on $\mathscr{B}$

$\Longrightarrow K_{\text{info}}$?

Jeffrey's Rule for choice functions



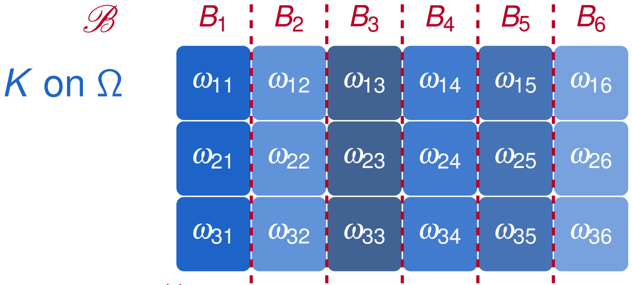
info: $\check{K}$ consisting of sets of gambles on $\mathscr{B}$

Find the smallest coherent K_{info} such that

$K_{\text{info}} \supseteq \check{K}$ [**agreeing on $\mathscr{B}$**]

$K_{\text{info}}\rfloor B \supseteq K\rfloor B$ for all $B \in \mathscr{B}$ [**rigidity**]

Jeffrey's Rule for choice functions



info: $\check{K}$ consisting of sets of gambles on $\mathscr{B}$

Find the smallest coherent K_{info} such that

$K_{\mathrm{info}} \supseteq \check{K}$ [**agreeing on $\mathscr{B}$**]

$K_{\mathrm{info}}\rfloor B \supseteq K\rfloor B$ for all $B \in \mathscr{B}$ [**rigidity**]

Theorem:

$$\mathrm{Rs}\Big(\mathrm{Posi}\Big(\check{K} \cup \bigcup_{B\in\mathscr{B}} \mathbb{I}_B(K\rfloor B)\Big)\Big)$$

is the unique smallest coherent K_{info} that satisfies **agreeing on $\mathscr{B}$** and **rigidity**.

Jeffrey's Rule for non-additive measures

Is there a version of Jeffrey's Rule for non-additive measures?

Jeffrey's Rule for non-additive measures

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Given:

a class $\mathscr{C}$ of lower probabilities

a lower probability $\underline{P} \in \mathscr{C}$ on Ω

info: a lower probability $\check{\underline{P}} \in \mathscr{C}$ on $\mathscr{B}$

Question: Is there a smallest $\underline{P}_{\text{info}} \in \mathscr{C}$ such that

$\underline{P}_{\text{info}}(B) \geq \check{\underline{P}}(B)$ for all $B \in \mathscr{B}$ [**agreeing on $\mathscr{B}$**]

$\underline{P}_{\text{info}}(A|B) \geq \underline{P}(A|B)$ for all $B \in \mathscr{B}, A \subseteq \Omega$ [**rigidity**] ?

Jeffrey's Rule for non-additive measures

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$\underline{P}_{\text{info}}(A|B) \geq \underline{P}(A|B)$ for all $B \in \mathscr{B}, A \subseteq \Omega$ [**rigidity**] ?

We study this question for **minitive measures**, **linear-vacuous models**, **pari-mutuel models** and **total variation models**.

Come and see our poster for answers!

A Study of Jeffrey's Rule With Imprecise Probability Models

1 The setting

Given You have a finite possibility space Ω and a probability measure P on Ω.

New information You observe a new probability measure $\check{P}$ on a partition $\mathscr{B}$ of Ω.

Question How should you update your probability measure P taking into account this information? We are looking for a probability measure $\widehat{P}$ on Ω that satisfies the constraints

- $\widehat{P}(B) = \check{P}(B)$ for all B in $\mathscr{B}$, [agreeing on $\mathscr{B}$]
- $\widehat{P}(A|B) = P(A|B)$ for all B in $\mathscr{B}$ and $A \subseteq \Omega$. [rigidity]

Jeffrey's Rule The unique probability measure $\widehat{P}$ on Ω that satisfies 'agreeing on $\mathscr{B}$' and 'rigidity' is given by

$$\widehat{P}(A) = \sum_{B \in \mathscr{B}} P(A|B)\check{P}(B) \qquad \text{for all } A \subseteq \Omega.$$

2 Sets of desirable gambles

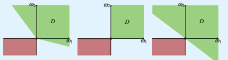
A gamble on Ω is a real-valued map on Ω. It is interpreted as an uncertain reward: if you have f then your capital changes by $f(\omega)$ when $\omega \in \Omega$ is determined.

Desirability A set of desirable gambles D is a set of gambles that the subject prefers over 0.

$f \in D$ means: the subject prefers f over 0.

Rationality axioms A set of desirable gambles D is coherent if for all gambles f and g and all real $\lambda > 0$:

D_1. $0 \notin D$; [avoiding null gain]
D_2. if $0 < f$ then $f \in D$; [desiring partial gain]
D_3. if $f \in D$ then $\lambda f \in D$; [positive scaling]
D_4. if $f, g \in D$ then $f + g \in D$. [combination]



a generic D — the vacuous D — a precise D

Conditioning Given a non-empty event $B \subseteq \Omega$, the conditional set of desirable gambles is

$$D\rfloor B = \{f \in \mathscr{L}(B) : \mathbb{I}_B f \in D\}.$$

Jeffrey's Rule You have a coherent set of desirable gambles D on Ω, and observe a new $\check{D}$ on the partition $\mathscr{B}$: $\check{D}$ contains gambles that are constant on the elements of $\mathscr{B}$. We are looking for a coherent set of desirable gambles $\widehat{D}$ on Ω that satisfies the constraints

- $\widehat{D} \supseteq \check{D}$, [agreeing on $\mathscr{B}$]
- $\widehat{D}\rfloor B \supseteq D\rfloor B$ for all B in $\mathscr{B}$. [rigidity]

It follows from [*G. de Cooman and F. Hermans. Imprecise probability trees: Bridging two theories of imprecise probability. Artificial Intelligence, 2008*] that there is a unique smallest coherent $\widehat{D}$ that satisfies 'agreeing on $\mathscr{B}$' and 'rigidity'. It is given by

$$\widehat{D} = \operatorname{posi}\Big(\check{D} \cup \textstyle\bigcup_{B \in \mathscr{B}} \mathbb{I}_B (D\rfloor B)\Big).$$

3 Sets of desirable gamble sets

$\mathscr{Q}(\Omega)$ is the collection of finite subsets of gambles on Ω. A set of desirable gamble sets $K \subseteq \mathscr{Q}$ is a collection of sets F of gambles that contain at least one gamble $f \in F$ that is preferred over 0.

$F \in K$ means: F contains at least one gamble that the subject prefers over 0.

So a set of desirable gamble sets can express more general types of uncertainty. It is equivalent to a choice function: $F \in K \Leftrightarrow 0 \notin C(\{0\} \cup F)$. [*T. Seidenfeld et al., Coherent choice functions under uncertainty. Synthese 2010*]

Rationality axioms A set of desirable gamble sets $K \subseteq \mathscr{Q}$ is coherent if for all F, F_1 and F_2 in $\mathscr{Q}$ and all $\{\lambda_{f,g}, \mu_{f,g} : f \in F_1, g \in F_2\} \subseteq \mathbb{R}$:

K_0. $\emptyset \notin K$;
K_1. $F \in K \Rightarrow F \setminus \{0\} \in K$;
K_2. $\{f\} \in K$, for all f in $\mathscr{L}_{>0}$;
K_3. if $F_1, F_2 \in K$ and if, for all f in F_1 and g in F_2, $(\lambda_{f,g}, \mu_{f,g}) > 0$, then

$$\{\lambda_{f,g} f + \mu_{f,g} g : f \in F_1, g \in F_2\} \in K;$$

K_4. if $F_1 \in K$ and $F_1 \subseteq F_2$ then $F_2 \in K$.

Here $\lambda_{1:n} := (\lambda_1, \dots, \lambda_n) > 0$ means '$\lambda_k \geq 0$ for all k, and $\lambda_\ell > 0$ for at least one ℓ'.

Representation For any coherent set of desirable gambles D, let $K_D := \{F \in \mathscr{Q} : F \cap D \neq \emptyset\}$ be the set of desirable gamble sets that represents Walley–Sen maximality.
A set of desirable gamble sets K is coherent if and only if there is a non-empty representing set of coherent sets of desirable gambles $\mathbf{D}$ such that $K = \bigcap_{D \in \mathbf{D}} K_D$, and the largest such set is $\mathbf{D}(K) := \{D : K \subseteq K_D\}$.
[*J. De Bock and G. de Cooman. Interpreting, axiomatising and representing coherent choice functions in terms of desirability. ISIPTA 2019*]

Conditioning Given a non-empty event $B \subseteq \Omega$, the conditional set of desirable gamble sets is

$$K\rfloor B = \{F \in \mathscr{Q}(B) : \mathbb{I}_B F \in K\}.$$

Jeffrey's Rule You have a coherent set of desirable gamble sets K on Ω, and observe a new $\check{K}$ on the partition $\mathscr{B}$. We are looking for a coherent set of desirable gamble sets $\widehat{K}$ on Ω that satisfies the constraints

- $\widehat{K} \supseteq \check{K}$, [agreeing on $\mathscr{B}$]
- $\widehat{K}\rfloor B \supseteq K\rfloor B$ for all B in $\mathscr{B}$. [rigidity]

There is a unique smallest coherent $\widehat{K}$ that satisfies 'agreeing on $\mathscr{B}$' and 'rigidity'. It is given by

$$\widehat{K} = \operatorname{Rs}\Big(\operatorname{Posi}\Big(\check{K} \cup \textstyle\bigcup_{B \in \mathscr{B}} \mathbb{I}_B (K\rfloor B)\Big)\Big).$$

4 Example: combination of two decision rules

finite set of pmfs $\mathscr{M} \subseteq \operatorname{int}(\Sigma_\Omega)$
The agent uses **maximality**:
$K = \{F : (\exists f \in F) \min_{p \in \mathscr{M}} E_p(f) > 0\}$

finite set of pmfs $\check{\mathscr{M}} \subseteq \operatorname{int}(\Sigma_{\mathscr{B}})$
The agent uses **E-admissibility**:
$\check{K} = \{F : (\forall p \in \check{\mathscr{M}})(\exists f \in F) E_p(f) > 0\}$

Can we update $\mathscr{M}$ using the new information $\check{\mathscr{M}}$, even if we use different decision rules?

Use Jeffrey's Rule for sets of desirable gamble sets.

In general, the result $\widehat{K}$ of Jeffrey's Rule is represented by

$$\widehat{\mathbf{D}} := \Big\{\operatorname{posi}\Big(\check{D} \cup \bigcup_{B \in \mathscr{B}} \mathbb{I}_B (D\rfloor B)\Big) : \check{D} \in \mathbf{D}(\check{K}), D \in \mathbf{D}(K)\Big\}.$$

In the present context, this representation is simplified as

$$\Big\{\operatorname{posi}\Big(D_{\check{p}} \cup \bigcup_{B \in \mathscr{B}} \mathbb{I}_B (D_{\mathscr{M}}\rfloor B)\Big) : \check{p} \in \check{\mathscr{M}}\Big\}$$

and as a consequence

$$F \in \widehat{K} \Leftrightarrow (\forall \check{p} \in \check{\mathscr{M}})(\exists f \in F) E_{\check{p}}\big(\min_{p \in \mathscr{M}} E_p(f|\mathscr{B})\big) > 0.$$

Combination of maximality and E-admissibility

5 Special cases: Jeffrey's Rule for non-additive measures

Is there a version of Jeffrey's Rule for non-additive measures?

Consider a special class $\mathscr{C}$ of coherent lower probabilities $\underline{P}$. We lift the domain of $\underline{P}$ to gambles f: $\underline{P}(f) := \min\{E(f) : (\forall A \subseteq \Omega) E(\mathbb{I}_A) \geq \underline{P}(A)\}$.
You have a lower probability $\underline{P} \in \mathscr{C}$ on Ω, and observe a new lower probability $\check{\underline{P}} \in \mathscr{C}$ on $\mathscr{B}$. You are looking for the least informative lower probability $\widehat{\underline{P}} \in \mathscr{C}$ such that

- $\widehat{\underline{P}}(B) \geq \check{\underline{P}}(B)$, [agreeing on $\mathscr{B}$]
- $\widehat{\underline{P}}(A|B) \geq \underline{P}(A|B)$, [rigidity]

for every $A \subseteq \Omega$ and B in $\mathscr{B}$.

Proposition. *Consider $\widehat{\underline{P}} \in \mathscr{C}$. Then $\widehat{\underline{P}}$ satisfies 'agreeing on $\mathscr{B}$' and 'rigidity' iff $\widehat{\underline{P}}(f) \geq \check{\underline{P}}(\underline{P}(f|\mathscr{B}))$ for every gamble f.*

So in order to answer the question, equivalently: check whether $\check{\underline{P}}(\underline{P}(\bullet|\mathscr{B}))$ belongs to $\mathscr{C}$.

Minitive measures Assume that $\mathscr{C}$ is the class of minitive measures $\underline{P}$:

$\underline{P}(A \cap B) = \min\{\underline{P}(A), \underline{P}(B)\}$ *minitivity on events*

$\underline{P}(\min\{f, g\}) = \min\{\underline{P}(f), \underline{P}(g)\}$ *minitivity on gambles*

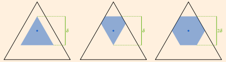
Proposition. *a) If $\underline{P}$ and $\check{\underline{P}}$ are minitive on gambles, then so is $\check{\underline{P}}(\underline{P}(\bullet|\mathscr{B}))$.*
b) If $\underline{P}$ or $\check{\underline{P}}$ is minitive on gambles, then $\check{\underline{P}}(\underline{P}(\bullet|\mathscr{B}))$ is minitive on events.
c) If $\underline{P}$ nor $\check{\underline{P}}$ is minitive on gambles, then $\check{\underline{P}}(\underline{P}(\bullet|\mathscr{B}))$ may not be minitive on events.

Distortion models Assume that $\mathscr{C}$ is either one of the classes of $\underline{P}$ that satisfy, for all $A \neq \Omega$:

$\underline{P}(A) = (1-\delta)P(A)$ *Linear-Vacuous model*

$\overline{P}(A) = \min\{1, (1+\delta)P(A)\}$ *Pari-Mutuel model*

$\underline{P}(A) = \max\{P(A) - \delta, 0\}$ *Total Variation model*



Proposition. *For any of the three classes $\mathscr{C}$ of lower probabilities mentioned above: if P and $\check{P}$ belong to $\mathscr{C}$, then $\check{\underline{P}}(\underline{P}(\bullet|\mathscr{B}))$ may not belong to $\mathscr{C}$.*

Universidad de Oviedo
Enrique Miranda
Department of Statistics and Operations Research, University of Oviedo, Spain

Arthur Van Camp
University of Bristol
Department of Philosophy, University of Bristol, UK