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Depth Functions for Partial Orders with a Descriptive Analysis of Machine Learning Algorithms

ISIPTA 2023

Working Group

Working Group *Foundations of Statistics and their Applications* of Prof. Dr. Thomas Augustin.
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with a Descriptive Analysis of Machine Learning Algorithms

Depth Function

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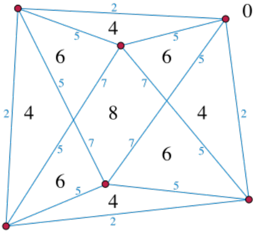

Figure: Simplicial Depth
(see https://en.wikipedia.org/wiki/Simplicial_depth, visited: 20.10.21)

Depth Function

Depth Functions measure **centrality** and **outlingless** of a data point with respect to a data cloud or an underlying distribution.

Idea: Adaptation of the simplicial depth to the set of partial orders

Approach: The representation of the simplicial depth via the convex closure system

Result: *union-free generic (ufg) depth function*

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Comparison of Machine Learning Algorithms

- Data Sets: 80 classification problems from OpenML.
- ML Algorithms: Random Forests (RF), Decision Tree (CART), Logistic regression (LR), L1-penalized logistic regression (Lasso) and k-nearest neighbours(KNN).
- Performance Measures: area under the curve, F-score, predictive accuracy and Brier score.

$\Rightarrow$ We obtain 80 posets

Comparison of Machine Learning Algorithms

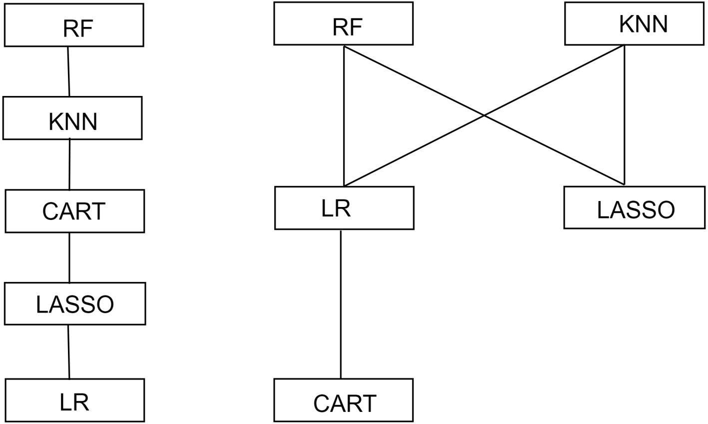


Figure: Example of two posets obtained by comparing five ml algorithms based on four performance measures. Each poset describes the performance based on one data set.

Minimal and Maximal Depth Value

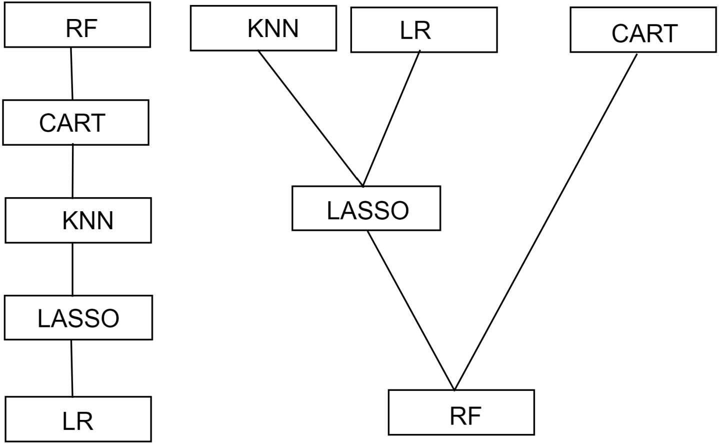
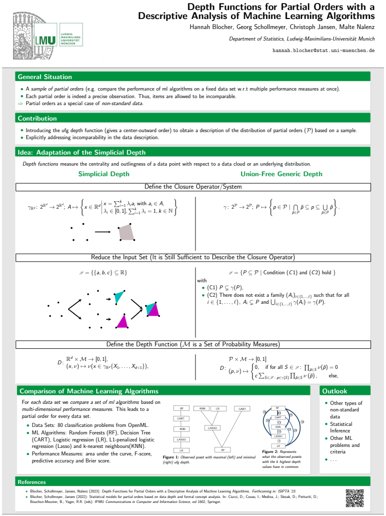


Figure: Observed poset with maximal (left) and minimal (right) ufg depth.

Poster and Paper



Depth Functions for Partial Orders with a Descriptive Analysis of Machine Learning Algorithms

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General Situation

- A *sample of partial orders* (e.g. compare the performance of ml algorithms on a fixed data set w.r.t multiple performance measures at once).
- Each partial order is indeed a precise observation. Thus, items are allowed to be incomparable.

⇒ Partial orders as a special case of *non-standard data*.

Contribution

- Introducing the ufg depth function (gives a center-outward order) to obtain a description of the distribution of partial orders ($\mathcal{P}$) based on a sample.
- Explicitly addressing incomparability in the data description.

Idea: Adaptation of the Simplicial Depth

Depth functions measure the centrality and outlingness of a data point with respect to a data cloud or an underlying distribution.

Simplicial Depth	Union-Free Generic Depth
Define the Closure Operator/System	
$\gamma_{\mathbb{R}^d}\colon 2^{\mathbb{R}^d} \to 2^{\mathbb{R}^d};\ A \mapsto \left\{ x \in \mathbb{R}^d \,\middle\vert\, \begin{array}{l} x = \sum_{i=1}^k \lambda_i a_i \text{ with } a_i \in A, \\ \lambda_i \in [0,1], \sum_{i=1}^k \lambda_i = 1, k \in \mathbb{N} \end{array} \right\}$	$\gamma\colon 2^{\mathcal{P}} \to 2^{\mathcal{P}};\ P \mapsto \left\{ p \in \mathcal{P} \mid \bigcap_{\tilde{p} \in P} \tilde{p} \subseteq p \subseteq \bigcup_{\tilde{p} \in P} \tilde{p} \right\}.$
Reduce the Input Set (It is Still Sufficient to Describe the Closure Operator)	
$\mathscr{S} = \{\{a, b, c\} \subseteq \mathbb{R}\}$	$\mathscr{S} = \{P \subseteq \mathcal{P} \mid$ Condition $(C1)$ and $(C2)$ hold $\}$ with: (C1) $P \subsetneq \gamma(P)$, (C2) There does not exist a family $(A_i)_{i \in \{1,\dots,\ell\}}$ such that for all $i \in \{1, \dots, \ell\}$, $A_i \subsetneq P$ and $\bigcup_{i \in \{1,\dots,\ell\}} \gamma(A_i) = \gamma(P)$.
Define the Depth Function ($\mathcal{M}$ is a Set of Probability Measures)	
$D\colon \begin{array}{l} \mathbb{R}^d \times \mathcal{M} \to [0,1], \\ (x, \nu) \mapsto \nu(x \in \gamma_{\mathbb{R}^d}\{X_1, \dots, X_{d+1}\}), \end{array}$	$D\colon \begin{array}{l} \mathcal{P} \times \mathcal{M} \to [0,1] \\ (p, \nu) \mapsto \begin{cases} 0, & \text{if for all } S \in \mathscr{S}\colon \prod_{\tilde{p} \in S} \nu(\tilde{p}) = 0 \\ c \sum_{S \in \mathscr{S}\colon p \in \gamma(S)} \prod_{\tilde{p} \in S} \nu(\tilde{p}), & \text{else,} \end{cases} \end{array}$

Comparison of Machine Learning Algorithms

For *each data set* we *compare a set of ml algorithms* based on *multi-dimensional performance measures*. This leads to a partial order for every data set.

- Data Sets: 80 classification problems from OpenML.
- ML Algorithms: Random Forests (RF), Decision Tree (CART), Logistic regression (LR), L1-penalized logistic regression (Lasso) and k-nearest neighbours(KNN).
- Performance Measures: area under the curve, F-score, predictive accuracy and Brier score.

Figure 1: *Observed poset with maximal (left) and minimal (right) ufg depth.*

Figure 2: *Represents what the observed posets with the k highest depth values have in common.*

Outlook

- Other types of non-standard data
- Statistical Inference
- Other ML problems and criteria
- ...

References

- Blocher, Schollmeyer, Jansen, Nalenz (2023): Depth Functions for Partial Orders with a Descriptive Analysis of Machine Learning Algorithms. *Forthcoming in: ISIPTA '23.*
- Blocher, Schollmeyer, Jansen (2022): Statistical models for partial orders based on data depth and formal concept analysis. In: Ciucci, D.; Couso, I.; Medina, J.; Slezak, D.; Petturiti, D.; Bouchon-Meunier, B.; Yager, R.R. (eds): IPMU *Communications in Computer and Information Science*, vol 1602, Springer.

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References

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