A modal logic for uncertainty: a completeness theorem

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Uncertainty Measures
- *Probabilities*
- *Belief functions*
- *Lower probabilities*

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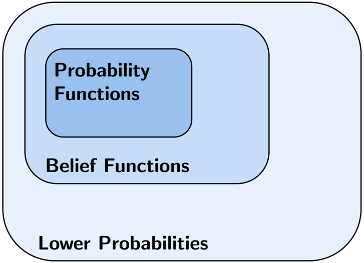
Modal Logic
- $S5 \rightarrow \Box \; \Diamond$
- P
- Ł

- Comparison of the measures
- Introduction of intermediate uncertainty measures

Uncertainty Measures



Finite Boolean Algebra: $\mathbf{A} = (A, \wedge, \vee, \neg, \bot, \top)$.

- A *probability function* P on an algebra $\mathbf{A}$ is a $[0,1]$-valued map satisfying:

  (P1) $P(\top) = 1$, $P(\bot) = 0$;

  (P2) $P(\theta \vee \phi) = P(\theta) + P(\phi)$, if $\theta \wedge \phi = \bot$. (Finite additivity)

Uncertainty Measures/II

- A *belief function* B on an algebra $\mathbf{A}$ is a $[0,1]$-valued map satisfying:

(B1) $B(\top) = 1$, $B(\bot) = 0$;

(B2) $B\left(\bigvee_{i=1}^{n} \psi_i\right) \geqslant \sum_{i=1}^{n} \sum_{\{J \subseteq \{1,\dots,n\} : |J| = i\}} (-1)^{i+1} B\left(\bigwedge_{j \in J} \psi_j\right)$ for $n = 1, 2, 3, \dots$. (∞-monotonicity)

Alternative Definition

- A *lower probability function* $\underline{P}$ on an algebra $\mathbf{A}$ is a $[0,1]$-valued map satisfying:

($\underline{P}$1) $\underline{P}(\top) = 1$, $\underline{P}(\bot) = 0$;

($\underline{P}$2) $\underline{P}(\theta \vee \phi) \geqslant \underline{P}(\theta) + \underline{P}(\phi)$, if $\theta \wedge \phi = \bot$. (Super-additivity)

Alternative Definitions

Łukasiewicz Logic

Standard MV-algebra: $[0,1]_{MV} = ([0,1], \oplus, \neg, 1)$

MV-Algebra System for Ł

$[0,1]$ is the real-unit interval

Strong disjunction: $a \oplus b = \min\{1, a+b\}$
Strong conjunction: $a \odot b = \max\{0, a+b-1\}$
Implication: $a \to b = \min(1-a+b, 1)$
Negation: $\neg a = 1-a$

Language

$\mathscr{L}$: language of **Ł** over n propositional variables

$\Box$ and P : two unary modalities

(CF) Classical Formulas

$$\varphi,\ \psi,\ \top,\ \bot,\ \varphi \wedge \psi,\ \varphi \vee \psi,\ \neg\varphi$$

(CMF) Classical Modal Formulas: closure of **(CF)** by $\Box$

$$\varphi \to \Box\psi,\ \Box\varphi \wedge \Diamond\psi,\ \Box(\Box\varphi \wedge \Diamond\psi)$$

(PMF) Probabilistic Modal Formulas: closure by P and **Ł**

- Atomic Probabilistic Formulas: $P(\varphi \to \Box\psi)$
- Compound Probabilistic Formulas: $P(\varphi) \to P(\Box\psi)$

$$\Box\varphi \to P(\psi),\ P(P(\varphi) \to P(\Box\psi))$$

(UMF) Uncertain Modal Formulas: smallest set that contains **(PMF)**, it is closed under $\Box$ and **Ł**

$$\Box P(\varphi),\ \Box P(\varphi) \to P(\Diamond\psi),\ \Box(P(\varphi) \to P(\Diamond\psi)),$$
$$\Box(\Box P(\varphi) \to P(\Diamond\psi))$$

Uniform Uncertain Model

$e_w : CF \to \{0, 1\}$
classical evaluation

$$U = (W,\ R,\ \{e_w\}_{w \in W},\ \{p_w\}_{w \in W})$$

S5-Kripke frame

S5 Axioms

p_w probability
distribution on W

Semantics

(CF) $\|\varphi\|_{\mathcal{U},w} = e_w(\varphi)$

(CMF) $\|\Box\psi\|_{\mathcal{U},w} = \inf\{\|\psi\|_{\mathcal{U},w'} \mid wRw'\}$

(PMF)
$$\begin{aligned}\|P(\psi)\|_{\mathcal{U},w} &= \sum\{p_w(w') \mid \|\psi\|_{\mathcal{U},w'} = 1\} \\ &= \sum\{p_w(w') \mid e_{w'}(\psi) = 1\}\end{aligned}$$

$$\|P(\Box\psi)\|_{\mathcal{U},w} = \sum\{p_w(w') \mid \|\Box\psi\|_{\mathcal{U},w'} = 1\}$$

(UMF) $\|\Box P(\psi)\|_{\mathcal{U},w} = \inf\{\|P(\psi)\|_{\mathcal{U},w'} \mid wRw'\}$

Example

p, q, r: propositional variables

$w_1 \models p, q, r$

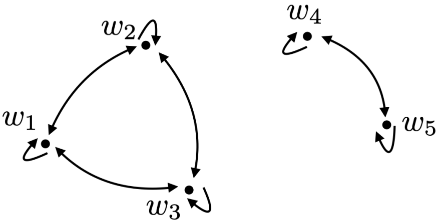
$w_2 \models p, \neg q, r$

$w_3 \models \neg p, q, \neg r$

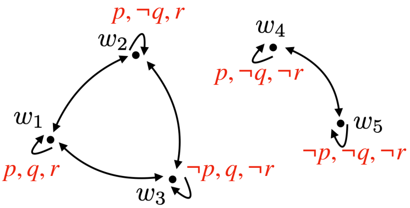
$w_4 \models p, \neg q, \neg r$

$w_5 \models \neg p, \neg q, \neg r$

	w_1	w_2	w_3	w_4	w_5
p_1	1/5	1/5	1/5	1/5	1/5
p_2	1/3	1/3	1/3	0	0
p_3	0	1/4	1/4	1/2	0
p_4	0	1/3	0	1/3	1/3
p_5	1/4	1/4	0	1/4	1/4



Example/II



$\mathcal{U} = (W, R, \{e_1, \ldots, e_5\}, \{p_1, \ldots, p_5\})$

$\varphi = r \to (p \wedge q)$

$\|P(\varphi)\|_{\mathcal{U},w_i} = p_i(w_1) + p_i(w_3) + p_i(w_4) + p_i(w_5)$

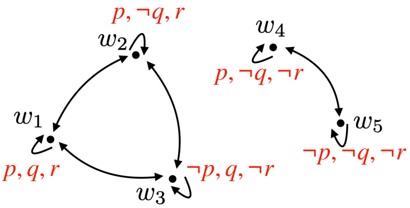
$\|P(\varphi)\|_{\mathcal{U},w_1} = 4/5$ and $\|P(\varphi)\|_{\mathcal{U},w_2} = 2/3$

$\|P(\Box\varphi)\|_{\mathcal{U},w_i} = p_i(w_4) + p_1(w_5)$

$\|P(\Box\varphi)\|_{\mathcal{U},w_1} = 2/5$ and $\|P(\Box\varphi)\|_{\mathcal{U},w_2} = 0$

$\|P(\Box\varphi)\|_{\mathcal{U},w_i} \leqslant \|P(\varphi)\|_{\mathcal{U},w_i}$ for all $i \Rightarrow P(\Box\varphi) \to P(\varphi)$ is valid in $\mathcal{U}$

Example/III



$$\mathcal{U} = (W, R, \{e_1, \dots, e_5\}, \{p_1, \dots, p_5\})$$

$$\varphi = r \to (p \wedge q)$$

$$\|\Box P(\varphi)\|_{\mathcal{U},w_i} = \min\{\|P(\varphi)\|_{\mathcal{U},w_j} \mid w_i R w_j\}$$

$$\begin{aligned}\|\Box P(\varphi)\|_{\mathcal{U},w_1} &= \min\{\|P(\varphi)\|_{\mathcal{U},w_1}, \|P(\varphi)\|_{\mathcal{U},w_2}, \|P(\varphi)\|_{\mathcal{U},w_3}\} = \\ &= \min\{4/5, 2/3, 3/4\} = 2/3\end{aligned}$$

R is reflexive $\Rightarrow \Box P(\varphi) \to P(\varphi)$ is valid in $\mathcal{U}$.
$x \to y$ and $z \to y$ imply, in **Ł**, $(x \wedge z) \to y$

$$(P(\Box\varphi) \wedge \Box P(\varphi)) \to P(\varphi)$$

holds in $\mathcal{U}$

System S5(FP(Ł))

(CPL) The axioms and rules of classical propositional logic for formulas in **CF**.
(S5) The axioms of S5 applied to **CMF**.
(FP(Ł)) Axioms and rules of FP(Ł) for **PMF** formulas, i.e. the axioms of Ł and the axioms for the modality P and Łukasiewicz implication:

$$(P1)\ P(\varphi \to \psi) \to (P(\varphi) \to P(\psi));$$
$$(P2)\ \neg P(\varphi) \equiv P(\neg\varphi);$$
$$(P3)\ P(\varphi \vee \psi) \equiv [(P(\varphi) \to P(\varphi \wedge \psi)) \to P(\psi)];$$
(NP) necessitation: from φ infer $P(\varphi)$.

(S5(Ł)) Axioms and rules of S5(Ł) for **UMF** formulas.

LF: the set of *lower probability formulas* is the smallest subset of **UMF** that contains all basic formulas of the form $\Box P(\varphi)$ for every classical formula φ and that is closed under Ł connectives.

Theorem

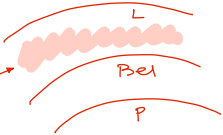
For every finite subset of formulas $T \cup \phi \subseteq (\mathbf{LF})$, the following conditions are equivalent:

(i) $T \vdash_{S5(FP(Ł))} \phi$

(ii) *for all finite (universal) S5 probability model* $\mathcal{U} = (W, R, \{e_w\}_{w \in W}, \{\mu_w\}_{w \in W})$ *with* $R = W \times W$, $\|\tau\|_{\mathcal{U}} = 1$ *for each* $\tau \in T$ *implies* $\|\phi\|_{\mathcal{U}} = 1$.

Uncertainty Measure	**Key Axiom**	**Modal Logic**
Probability	Additivity	$P(\varphi)$
Belief Functions	∞-monotone	$P(\Box\varphi)$
Lower Probabilities	Super-additivity	$\Box P(\varphi)$

- What is coherence?
- Can we interprete, e.g., k-capacities?



- Can we introduce *Lower Belief Functions*?
  - $\Box P(\Box \varphi)$

See you at the poster!

Thank you!

Definition (MV-Algebra)

An MV-Algebra is a structure of the form $A = (A, \oplus, \neg, 1)$ where A is a nonempty set, $\oplus$ is a binary and $\neg$ a unary operation on A, while 1 is a constant. $(A, \oplus, 1)$ is a commutative monoid, and the following equations are satisfied:

- $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$,
- $\neg\neg x = x$.

Definition (System for Ł)

A Hilbert-style axiomatisation for Łukasiewicz Logic is the following.

$$
\begin{aligned}
[Tr]: &\quad (\varphi \to \psi) \to ((\psi \to \xi) \to (\varphi \to \xi)) \\
[We]: &\quad \varphi \to (\psi \to \varphi) \\
[Ex]: &\quad (\varphi \to (\psi \to \xi)) \to (\psi \to (\varphi \to \xi)) \\
[\wedge\text{-}1]: &\quad (\varphi \wedge \psi) \to \varphi \\
[\wedge\text{-}2]: &\quad (\varphi \wedge \psi) \to \psi \\
[\wedge\text{-}3]: &\quad (\xi \to \varphi) \to ((\xi \to \psi) \to (\xi \to (\varphi \wedge \psi))) \\
[\vee\text{-}1]: &\quad \varphi \to (\varphi \vee \psi) \\
[\vee\text{-}2]: &\quad \psi \to (\varphi \vee \psi) \\
[\vee\text{-}3]: &\quad (\psi \to \varphi) \to ((\xi \to \varphi) \to ((\psi \vee \xi) \to \varphi)) \\
[Lin]: &\quad (\varphi \to \psi) \vee (\psi \to \varphi) \\
[\bot]: &\quad \bot \to \varphi \\
[Waj]: &\quad ((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)
\end{aligned}
$$

Definition (Belief Function)

BFs on boolean algebras can be characterized in terms of mass function m s.t.

- $m(\emptyset) = 0$
- $\sum_X m(X) = 1$

$$B(\psi) = \sum_{X \subseteq \{\alpha_i | \alpha_i \leqslant \psi\}} m(X)$$

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Definition (Lower Probability/II)

A *lower probability* $\underline{P}$ on an algebra **A** is a monotone $[0, 1]$-valued map satisfying:

(L1) $\underline{P}(\top) = 1$, $\underline{P}(\bot) = 0$;

(L2) For all natural numbers n, m, k and all $\psi_1, \ldots, \psi_n$, if $\{\{\psi_1, \ldots, \psi_n\}\}$ is an (m, k)-cover of $(\varphi, \top)$, then $k + m\underline{P}(\varphi) \geqslant \sum_{i=1}^{n} \underline{P}(\psi_i)$.

Definition (Lower Probability/III)

Let $\mathbb{P}$ be a set of probability functions on an algebra **A**. A *lower probability* $\underline{P}$ on **A** with respect to $\mathbb{P}$ is a $[0, 1]$-valued map defined as $\underline{P}(\psi) = \inf\{P(\psi) : P \in \mathbb{P}\}$.

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Definition

The axioms of $S5$ are the following:

(CPL) Axioms of classical propositional logic;
(K) $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$;
(T) $\Box\varphi \rightarrow \varphi$;
(4) $\Box\varphi \rightarrow \Box\Box\varphi$;
(B) $\varphi \rightarrow \Box\Diamond\varphi$.

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