Eliciting hybrid probability-possibility functions and their decision evaluation models

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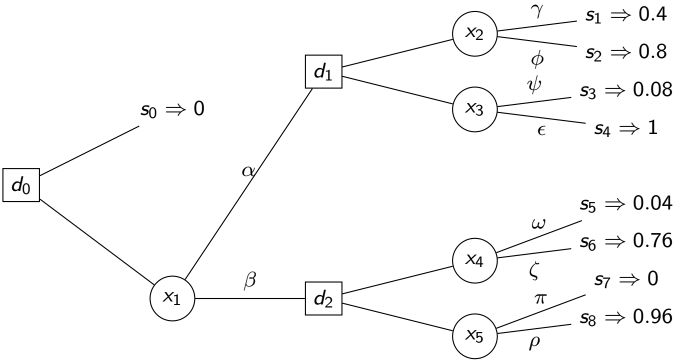
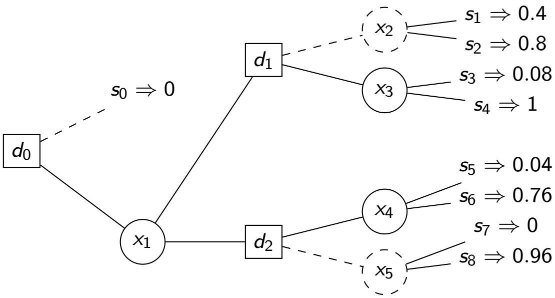
Outline of the presentation

Decision Tree



Decision Tree



Desirable assumptions

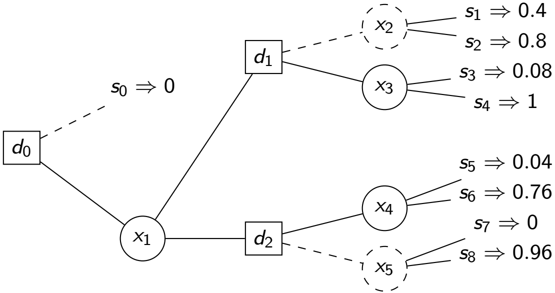
Three assumptions are desirable in order to accept an optimal strategy without questioning its meaning. Those assumptions are:

- *Dynamic Consistency*
- *Consequentialism*
- *Tree Reduction*

Dynamic Consistency

Definition (Dynamic Consistency)

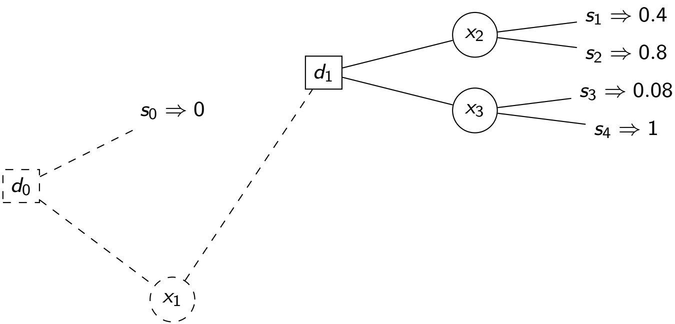
When reaching a decision node by following an optimal strategy, the best decision at this node is the one that had been considered so when computing this strategy, i.e. prior to applying it.



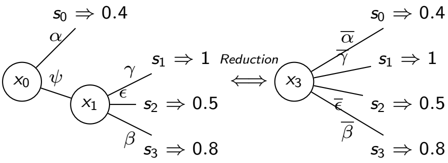
Consequentialism

Definition (Consequentialism)

The best decision at each step of the decision tree only depends on potential consequences at this point.



Tree Reduction



Fuzzy measures with lottery reduction property

We are interested on special fuzzy measures representable by means of lotteries having the lottery reduction property.
$\Longrightarrow$ We need decomposable fuzzy measures
$\Longrightarrow$ Hybrid possibility-probability measures are the only ones that possess the lottery reduction property.

Hybrid distributions generate a class of decomposable capacities

if $A \cap B = \emptyset$:

$$\rho^\alpha(A \cup B) = S^\alpha(\rho^\alpha(A), \rho^\alpha(B))$$

$$S^\alpha(x, y) = \begin{cases} \min(1, x + y - \alpha) \text{ if } x > \alpha, y > \alpha \\ \max(x, y) \text{ otherwise,} \end{cases}$$

In order to reduce probability-possibility lotteries, an operation $*$ is needed to generalize probabilistic independence. If A and B are disjoint sets independent of another set C:

$$\begin{aligned} \rho^\alpha((A \cup B) \cap C) &= S^\alpha(\rho^\alpha(A), \rho^\alpha(B)) * \rho^\alpha(C) \\ &= S^\alpha(\rho^\alpha(A) * \rho^\alpha(C), \rho^\alpha(B) * \rho^\alpha(C)). \end{aligned}$$

This distributivity property is valid only when the operation $*$ is a triangular norm of the form

$$x *_\alpha y = \begin{cases} \alpha + \frac{(x-\alpha)(y-\alpha)}{1-\alpha} \text{ if } x > \alpha, y > \alpha \\ \min(x, y) \text{ otherwise.} \end{cases}$$

Hybrid possibility-probability distribution

we will use convex combinations of possibility and probability distributions:

$$\rho^{\alpha}(s) = \alpha\pi(s) + (1-\alpha)p(s), \quad \alpha \in [0,1]$$

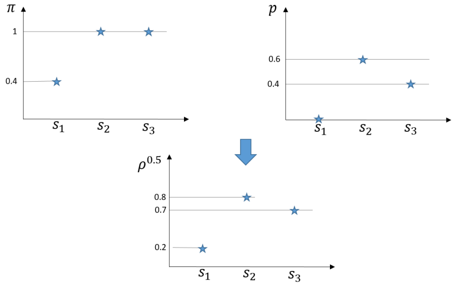
where p and π satisfy the constraint $p(s) = 0$ if $\pi(s) < 1$ for all s (see [DFGR21] for more details).

- $\Rightarrow$ ρ^{α} is a possibility distribution if $\alpha = 1$
- $\Rightarrow$ a probability distribution if $\alpha = 0$

As a consequence, note that $1 \leq \sum_{s \in S} \rho^{\alpha}(s) \leq n$.

The decision-maker defines probabilities over the fully possible states of nature ($\pi(s) = 1$), while more or less impossible ones are taken into account ($\pi(s) < 1$).

Example of hybrid distribution



$\rho^{0.5}(A)$

$\rho^{0.5}(\{s_1, s_2\}) = \max(0.2, 0.8) = 0.8$, $\rho^{0.5}(\{s_1, s_3\}) = \max(0.2, 0.7) = 0.7$,
$\rho^{0.5}(\{s_2, s_3\}) = 0.8 + 0.7 - 0.5 = 1$.
$\Rightarrow$ This distribution defines a convex set of probability distributions. We can express this probability set by inequalities:
$P(\{s_1, s_2, s_3\}) = 1$, $0 \leq P(\{s_1\}) \leq 0.2$, $0.3 \leq P(\{s_2\}) \leq 0.8$, $0.2 \leq P(\{s_2\}) \leq 0.7$,
$0.3 \leq P(\{s_1, s_2\}) \leq 0.8$, $0.2 \leq P(\{s_1, s_3\}) \leq 0.7$ and $0.3 \leq P(\{s_2, s_3\}) \leq 1$.

Hybrid model to interpret distribution of weights

Example

Consider two distributions on $S = \{a, b\}$

1: $\rho_a = \rho_b = 0.6$

2: $\rho'_a = \rho'_b = 0.5$

We can see that renormalizing these distributions in agreement with possibility or probability, the resulting two distributions 1 and 2 are the same:

- $p_a^1 = p_a^2 = 0.5 = p_b'^1 = p_b'^2$
- $\pi_a'^1 = \pi_a'^2 = 1 = \pi_b'^1 = \pi_b'^2$

Hence no distinction between 1 and 2 can be made using this kind of transformation.

Using the hybrid interpretation

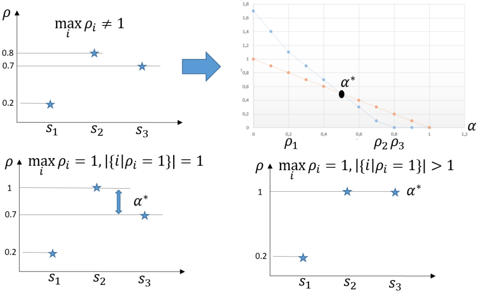
Case 1: with $\alpha = 0.2$, $\pi_a = \pi_b = 1$, *and* $p_a = p_b = 0.5$ *we can check that* $\rho_a = \rho_b = 0.2 + 0.8 \cdot 0.5$, *(a mixture between uniform probabilities and possibilities).*

Case 2: $\alpha = 0$, $\pi_a = \pi_b = 1$, *and* $p_a = p_b = 0.5 = \rho_a = \rho_b$, *(a pure probability distribution).*

Elicitation problem

The question is:

given a distribution of weights $(\rho_1, \ldots \rho_n) \in [0,1]^n$ on S such that $\sum_{i=1}^n \rho_i \geq 1$, does there exist a threshold $\alpha \in [0,1]$, a possibility distribution π and a probability distribution p on S, such that $\rho = \alpha\pi + (1-\alpha)p$? If yes, is the 3-tuple (α, π, p) uniquely defined?



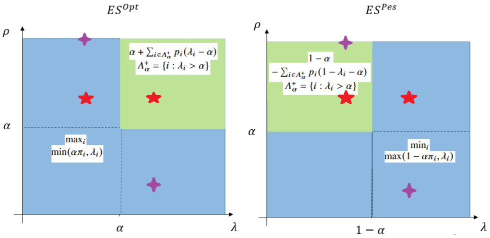
The utility of generalized lotteries

Example

Situation a has utility $\lambda_a = 0.3$ *and b* $\lambda_b = 0.7$. $\alpha = 0.5$

Decision 1: $\pi_a = \pi_b = 1$, *and* $p_a = p_b = 0.5$

Decision 2: $\pi_a = 1, \pi_b = 0.2$, *and* $p_a = 1, p_b = 0$.



$ES^{Opt}(D1) = 0.5 + 0.5 * (0.7 - 0.5) = 0.6 \succ_{ES^{Opt}} ES^{Opt}(D2) = max(min(0.5, 0.3), min(0.1, 0.7)) = 0.3$

$ES^{Pes}(D1) = 0.5 - 0.5(1 - 0.3 - 0.5) = 0.4 \succ_{ES^{Pes}} ES^{Pes}(D2) = 0.5 - (1 - 0.3 - 0.5) = 0.3$

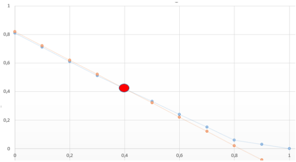
Elicitation from global ratings of loteries

The dataset is a set of tuples (π^j, p^j, β^j) $j \in J = \{1, ..., m\}$ where π^j is a possibility distribution, p^j is a probability distribution j is a strategy, and β^j is the global evaluation given by an expert.

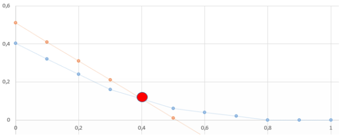
	$i:$	1	2	3	4	5	β
	λ_i	0.01	0.3	0.5	0.8	1	
$j=1$	π^1	0.2	0.6	1	1	0	0.8
	p^1	0	0	0	1	0	
$j=2$	π^2	1	1	0.5	0.5	0	0.3
	p^2	0.4	0.6	0	0	0	
$j=3$	π^3	0	1	0.5	1	1	0.82
	p^3	0	0.1	0	0.6	0.3	
$j=4$	π^4	1	1	1	1	1	0.51
	p^4	0.2	0.3	0.3	0.2	0	

	$\underline{\alpha}$	$\overline{\alpha}$
$j=1$	0	1
$j=2$	0.3	0.6
$j=3$	0	0.82
$j=4$	0	0.51
α	0.3	0.51

$j=3$



$j=4$



So, the decision-maker is consistent across all four examples: $\alpha = 0.4$ is a valid choice for the 4 items.

See you at the poster session!!

Références I

Didier Dubois, Hélène Fargier, Romain Guillaume, and Agnès Rico, *Sequential decision-making under uncertainty using hybrid probability-possibility functions*, Modeling Decisions for Artificial Intelligence (Cham) (Vicenç Torra and Yasuo Narukawa, eds.), Springer International Publishing, 2021, pp. 54–66.