A Comparison Between a Frequentist, Bayesian and Imprecise Bayesian Approach to Delay Time Maintenance

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I. Introduction

$Y = U + H$, with Y : **failure time**, U : **defect appearance time**, H : **delay time**.

$$f_Y(y) = k_f \lambda_h e^{-k_f y} \int_{h=0}^{y} e^{(k_f - \lambda_h)h} dh = \frac{k_f \lambda_h e^{-k_f y}}{k_f - \lambda_h} \left(e^{(k_f - \lambda_h)y} - 1 \right) \text{ when } \lambda_h \neq k_f.$$

We seek to minimise $D_1(T) = \frac{E(Downtime)}{E(Operating\ Time)}$ and $C_1(T) = \frac{E(Cost)}{E(Operating\ Time)}$.

$$E(Downtime) = p_{intact} d_i + p_{defect} (d_i + d_{defect}) + p_{failure} d_{br}$$

$$E(Operating\ Time) = E(Y | Y \leq T) F_Y(T) + T \left(1 - F_Y(T) \right)$$

$$E(Cost) = p_{intact} c_i + p_{defect}(c_i + c_{defect}) + p_{failure} c_{br}$$

II. Virtual case study

Pump where one defect progressively deteriorates into a failure.

k_f = 1/4 defects/month, λ_h = 9/100 failures/defect/month

However, the enterprise only knows that $\lambda_h \in [0.01; 0.1]$ failures/defect/month.

d_i = 0.4 days, d_{defect} = 2 days, d_{br} = 28 days, c_i = 110 €, c_{defect} = 1000 €, and c_{br} = 9000 €.

8 samples of failure times $S_1 \subset S_2 \subset S_3 \subset S_4 \subset S_5 \subset S_6 \subset S_7 \subset S_8$.

Sizes of the samples : 2, 5, 10, 20, 60, 200, 600, and 1000 .

Let $T_{D_1,opt,\lambda_{h,true}}$ be the optimal inspection period minimising $D_1(T)$ knowing $\lambda_{h,true}$

$T_{D_1,opt,S}$ is the optimal inspection period minimising $D_1(T)$ based on sample S.

The enterprise will use this type of pumps in **6** factories for a period equal to t = **4 years** of operational time.

Consequently, it is interesting to consider the quantities

$$\varDelta Downtime = 6\left(D_1(T_{D_1,opt,S}, \lambda_{h,true}) - D_1(T_{D_1,opt,\lambda_{h,true}}, \lambda_{h,true})\right)t$$

$$\varDelta Cost = 6\left(C_1(T_{C_1,opt,S}, \lambda_{h,true}) - C_1(T_{C_1,opt,\lambda_{h,true}}, \lambda_{h,true})\right)t$$

Suboptimality of the decision based on sample *S*.

III. Frequentist approach

Maximise the likelihood $L(\lambda_h | y_1, y_2, \ldots, y_n) = \prod_{i=1}^{n} f_Y(y_i) = \prod_{i=1}^{n} \frac{k_{f,0}\lambda_h}{k_{f,0} - \lambda_h} \left(e^{-\lambda_h y_i} - e^{-k_{f,0} y_i} \right)$.

Sample Size	$\lambda_{h,MLE}$	$T_{D_1,opt,S}$	$T_{C_1,opt,S}$	Δ *Downtime*	Δ *Cost* (€)
2	0.0662	60.3735	67.7731	3.2699	1428.5680
5	0.0690	58.3667	64.6834	2.4081	1033.6236
10	0.0960	45.6374	47.2540	0.1278	49.4582
20	0.0946	46.1062	47.8410	0.0766	29.7284
60	0.0860	49.3381	51.9885	0.0654	26.0579
200	0.0862	49.2551	51.8798	0.0589	23.4314
600	0.0880	48.5267	50.9301	0.0159	6.2836
1000	0.0902	47.6784	49.8361	0.0002	0.0605
$+\infty$	0.0900	47.7537	49.9327	0.0000	0.0000

Poor decisions for small sample sizes !

IV. Precise Bayesian approach

$$f(\lambda_h|S) = \frac{L(S|\lambda_h) f_{1,0}(\lambda_h)}{\int_{\lambda_h=0.01}^{0.1} L(S|\lambda_h) f_{1,0}(\lambda_h) d\lambda_h} \quad , \quad E(\lambda_h|S) = \int_{\lambda_h=0.01}^{0.1} \lambda_h f(\lambda_h|S) \quad .$$

Sample Size	$E(\lambda_h\|S)$	$T_{D_1,opt,S}$	$T_{C_1,opt,S}$
0	0.0550	72.1056	88.7999
2	0.0613	64.9950	75.5063
5	0.0664	60.5123	68.0865
10	0.0785	52.9380	56.8024
20	0.0829	50.7739	53.8654
60	0.0843	50.0010	52.9380
200	0.0863	49.2281	51.8559
600	0.0881	48.4552	50.9285
1000	0.0903	47.6823	49.8464
$+\infty$	0.0900	47.7537	49.9327

with a uniform prior.

$$E_{\lambda_h}\left(D_1(T,\lambda_h)\right)=\int_{\lambda_h=0.01}^{0.1} D_1(T,\lambda_h)f(\lambda_h|S)d\lambda_h$$ and $$E_{\lambda_h}\left(C_1(T,\lambda_h)\right)=\int_{\lambda_h=0.01}^{0.1} C_1(T,\lambda_h)f(\lambda_h|S)d\lambda_h$$

are minimised with respect to *T*.

Comparison with the frequentist approach

n_Y	$\Delta Downtime_B$	$\Delta Downtime_F$	$\Delta Cost_B$	$\Delta Cost_F$
2	5.5803	3.2699	2568.2310	1428.5680
5	3.3308	2.4081	1469.9209	1033.6236
10	0.6447	0.1278	261.8602	49.4582
20	0.2298	0.0766	91.3934	29.7284
60	0.1295	0.0654	54.4745	26.0579
200	0.0568	0.0589	22.8585	23.4314
600	0.0131	0.0159	6.2591	6.2836
1000	0.0001	0.0002	0.0481	0.0605
$+\infty$	0.0000	0.0000	0.0000	0.0000

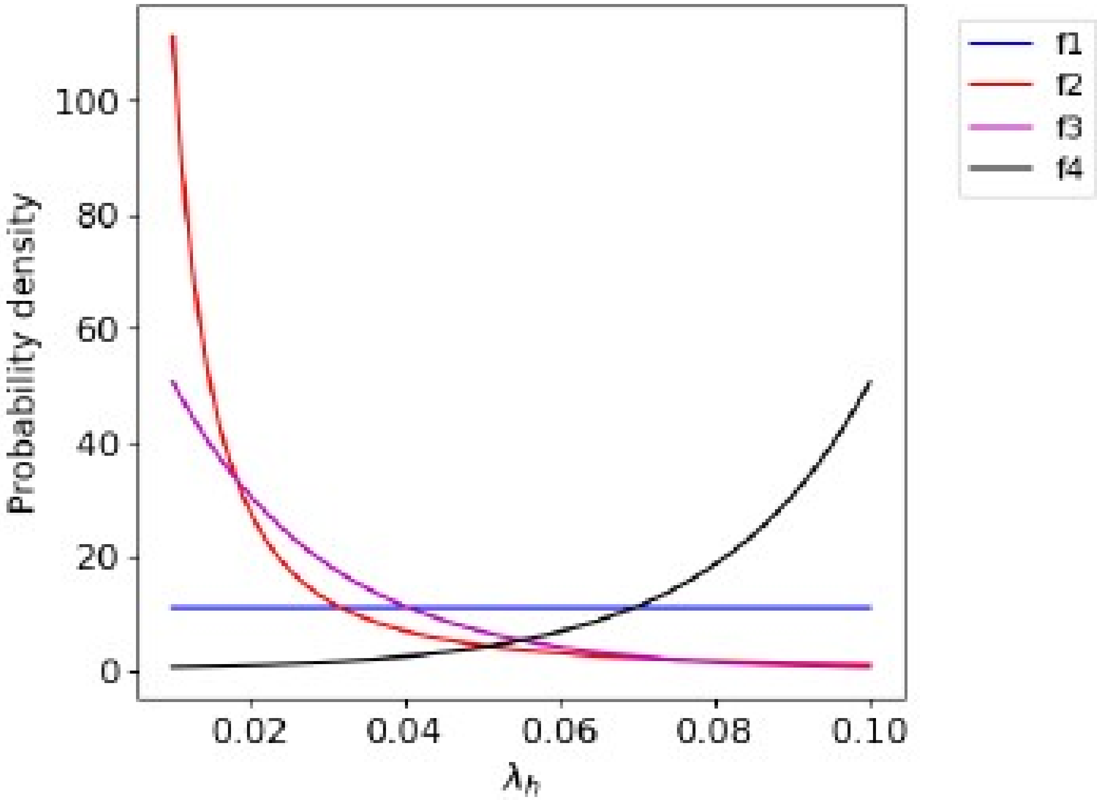
IV. Imprecise Bayesian approach

$f_{1,0}(\lambda_h)$: prior uniform with respect to $\lambda_h \in [0.01; 0.1]$

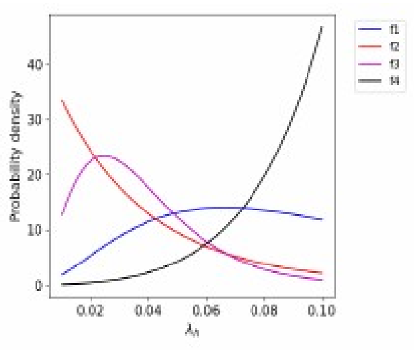
$f_{2,0}(\lambda_h) = \frac{1}{\lambda_h^2} \frac{1}{\frac{1}{\lambda_{h,min}} - \frac{1}{\lambda_{h,max}}}$: prior uniform with respect to $E(H) = \frac{1}{\lambda_h} \in [10; 100]$

$f_{3,0}(\lambda_h) = \frac{te^{-\lambda_h t}}{e^{-0.01t} - e^{-0.1t}}$: prior uniform with respect to $prop_{Safe}(t, \lambda_h) \in [e^{-0.1t}, e^{-0.01t}]$

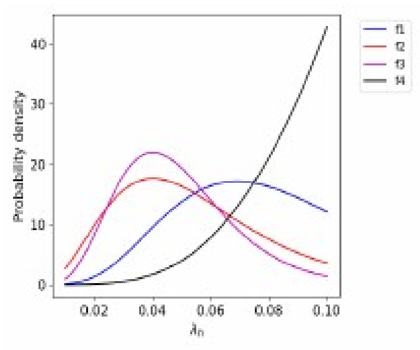
$f_{4,0}(\lambda_h) = t\frac{e^{\lambda_h t}}{e^{0.1t} - e^{0.01t}}$: prior uniform with respect to $\frac{1}{prop_{safe}(t, \lambda_h)} \in [e^{0.01t}; e^{0.1t}]$.



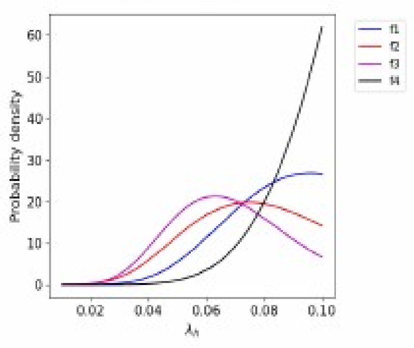
The four priors of λ_h.



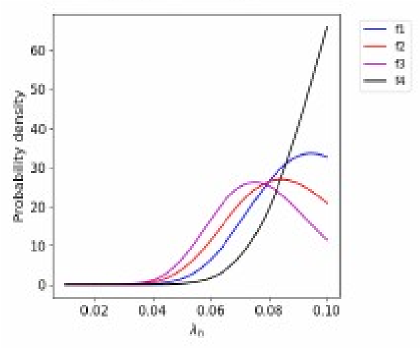
(a) Sample 1



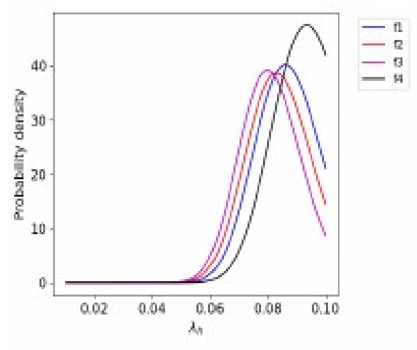
(b) Sample 2



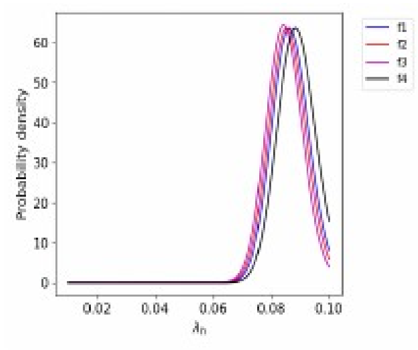
(c) Sample 3



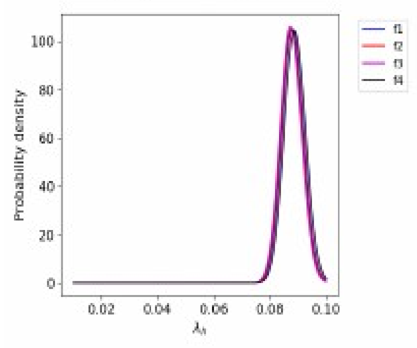
(d) Sample 4



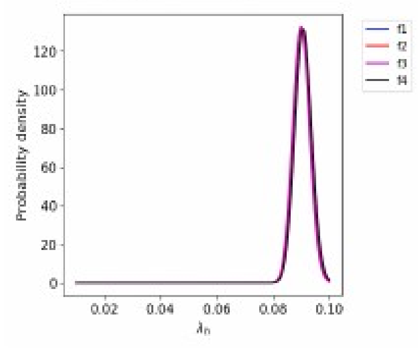
(a) Sample 5



(b) Sample 6



(c) Sample 7



(d) Sample 8

Posteriors

n_Y	$\lambda_{h,MLE}$	$E_1(\lambda_h)$	$min_i\Big(E_i(\lambda_h)\Big)$	$max_i\Big(E_i(\lambda_h)\Big)$
0		0.0550	0.0256	0.0810
2	0.0662	0.0613	0.0364	0.0812
5	0.0690	0.0664	0.0475	0.0816
10	0.0960	0.0785	0.0655	0.0867
20	0.0946	0.0829	0.0748	0.0883
60	0.0860	0.0843	0.0800	0.0879
200	0.0862	0.0863	0.0845	0.0881
600	0.0880	0.0881	0.0874	0.0889
1000	0.0902	0.0903	0.0899	0.0908

Expectation intervals for λ_h

n_Y	$min(T_{D_1,opt})$	$max(T_{D_1,opt})$	$\Delta Downtime_B$	$\Delta Downtime_F$
0	51.7014	$+\infty$	9.8058	
2	51.5468	118.6333	5.5803	3.2699
5	51.3922	83.0805	3.3308	2.4081
10	49.0735	61.1306	0.6447	0.1278
20	48.3007	54.9475	0.2298	0.0766
60	48.6098	52.0105	0.1295	0.0654
200	48.4552	50.0010	0.0568	0.0589
600	48.1461	48.7644	0.0131	0.0159
1000	47.5278	47.8369	0.0001	0.0002

Imprecise decision intervals

n_Y	$min(T_{C_1,opt})$	$max(T_{C_1,opt})$	$\Delta Cost_B$	$\Delta Cost_F$
0	55.1021	$+\infty$	4837.4707	
2	54.9475	$+\infty$	2568.2310	1428.5680
5	54.6383	114.7689	1469.9209	1033.6236
10	51.5468	69.0140	261.8602	49.4582
20	50.7739	59.5848	91.3934	29.7284
60	50.9285	55.5658	54.4745	26.0579
200	50.9285	52.7834	22.8585	23.4314
600	50.4647	51.2376	6.2591	6.2836
1000	49.5373	50.0010	0.0481	0.0605

V. Discussion and conclusions

Both frequentism and precise Bayesianism lead to poor decisions for small sample sizes.

Precise Bayesians could realise that by seeing that $var(T_{opt})$ is very large when there are few data.

Unfortunately, precise Bayesianism only consider the **expected value** while making decisions.

Imprecise Bayesianism is an improvement in that a lack of data corresponds to large decision intervals whereas plenty of data lead to small decision intervals.

It would be worthwhile to extend the imprecise Bayesian approach to (much) more complex delay time models in maintenance !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