

No-Arbitrage Pricing with α -DS Mixtures in a Market with Bid-Ask Spreads

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Our group and main active projects



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Castronovo



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Frezza



Silvia
Lorenzini



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Marconi

- NRRP project: “Future Artificial Intelligence Research”
- PRIN project: “Models for dynamic reasoning under partial knowledge to make interpretable decisions”

Classical “ideal” financial market

PERFECT MARKET HYPOTHESES:

Absence of frictions:

No taxes nor transaction costs;

Infinitely divisible securities;

Unlimited short sale;

⇒ Selling and buying prices of securities coincide

Competitiveness:

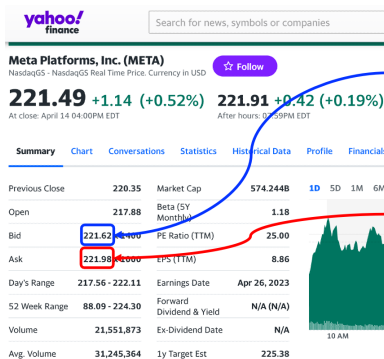
Profit maximizer agents;

Price-taker agents.

Classical no-arbitrage principle

Prices can be expressed as **discounted expectations** with respect to an “artificial” probability measure Q .

Is the absence of frictions hypothesis realistic?



Bid price
(if we want to SELL):
The highest price that a buyer is willing to pay.

Ask price
(if we want to BUY):
The lowest price that a seller is willing to accept.

OUR GOAL: Replace Q with a non-additive measure so as to consider bid-ask spreads.

α -DS mixtures

Consider:

- $\Omega = \{1, \dots, n\}$ with $n \geq 1$, a finite set of states of the world
- $\mathcal{P}(\Omega)$, power set of events

Definition

Let $\alpha \in [0, 1]$. A mapping $\varphi_\alpha : \mathcal{P}(\Omega) \rightarrow [0, 1]$ is called an α -**DS mixture** if there exists a belief function $Bel : \mathcal{P}(\Omega) \rightarrow [0, 1]$ with dual plausibility function Pl such that, for all $A \in \mathcal{P}(\Omega)$,

$$\begin{aligned}\varphi_\alpha(A) &= \alpha Bel(A) + (1 - \alpha) Pl(A) \\ &= \alpha Bel(A) + (1 - \alpha)(1 - Bel(A^c)).\end{aligned}$$

The belief function Bel is said to **represent** the α -DS mixture φ_α .

\implies The representation is unique when $\alpha \neq \frac{1}{2}$

α -DS mixture Choquet expectation

Every φ_α uniquely extends to a functional $\mathbb{C}_{\varphi_\alpha} : \mathbb{R}^\Omega \rightarrow \mathbb{R}$ by setting, for every $X \in \mathbb{R}^\Omega$,

$$\mathbb{C}_{\varphi_\alpha}[X] = \int X \, d\varphi_\alpha$$

Hurwicz-like representation:

$$\mathbb{C}_{\varphi_\alpha}[X] = \alpha \min_{P \in \mathcal{C}_{Bel}} \mathbb{E}_P[X] + (1 - \alpha) \max_{P \in \mathcal{C}_{Bel}} \mathbb{E}_P[X]$$

where \mathcal{C}_{Bel} is the core of Bel

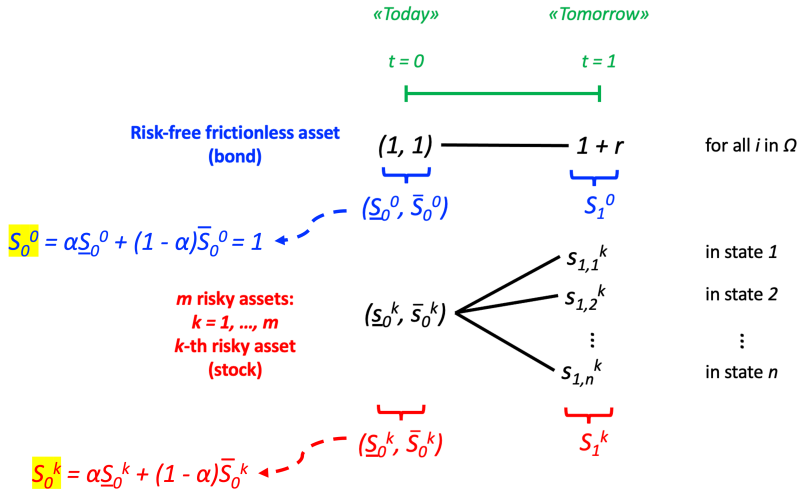
Möbius-like representation:

$$\mathbb{C}_{\varphi_\alpha}[X] = \sum_{B \in \mathcal{U}} [[X]]^\alpha(B) \mu(B)$$

where μ is the Möbius inverse of Bel and

$$\mathcal{U} = \mathcal{P}(\Omega) \setminus \{\emptyset\} \text{ and } [[X]]^\alpha(B) = \alpha \min_{i \in B} X(i) + (1 - \alpha) \max_{i \in B} X(i)$$

One-period market with bid-ask spreads



No-arbitrage pricing under α -PRU

Given a portfolio $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_m)^T \in \mathbb{R}^{m+1}$ we define:

Price at time $t = 0$: $V_0^\lambda = \lambda_0 + \sum_{k=1}^m \lambda_k S_0^k$

Payoff under α -PRU at time $t = 1$: $V_1^\lambda = \lambda_0(1+r) + \sum_{k=1}^m \lambda_k \llbracket S_1^k \rrbracket^\alpha$

α -PRU principle at time $t = 1$

PRU (Partially Resolving Uncertainty) due to Jaffray: An agent may only acquire that $B \neq \emptyset$ occurs, without knowing which is the true $i \in B$

α -pessimism: An agent always considers the α -mixture between the minimum and the maximum of random payoffs on every $B \neq \emptyset$

First and Second FTAP under α -PRU

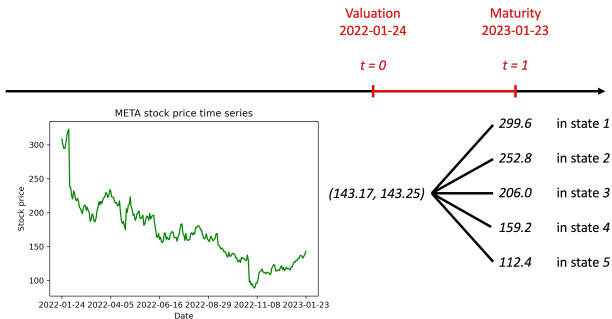
Theorem (First FTAP under α -PRU)

Let $\alpha \in [0, 1]$. The following conditions are equivalent:

- (i) there exists an α -DS mixture $\widehat{\varphi}_\alpha$ represented by a belief function strictly positive on \mathcal{U} and such that $\frac{C_{\widehat{\varphi}_\alpha}[S_1^k]}{1+r} = S_0^k$, for $k = 1, \dots, m$;
- (ii) for every $\lambda \in \mathbb{R}^{m+1}$ none of the following conditions holds:
- (a) $\underbrace{V_0^\lambda < 0}_{\text{we are paid today}}$ and $\underbrace{V_1^\lambda \geq 0 \text{ and } V_1^\lambda(\{i\}) = 0, \text{ for all } i;}_{\text{in all events we do not lose money and have a null gain on states tomorrow}}$
- (b) $\underbrace{V_0^\lambda \leq 0}_{\text{we do not pay today}}$ and $\underbrace{V_1^\lambda \geq 0 \text{ with } V_1^\lambda(\{i\}) > 0 \text{ for some } i.}_{\text{in all events we do not lose money and have a positive gain in a state tomorrow}}$

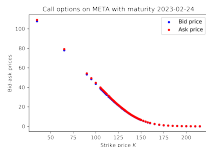
\implies The Second FTAP under α -PRU gives a sufficient condition (α -PRU completeness) for uniqueness of $\widehat{\varphi}_\alpha$

META stock market data with bid-ask spreads

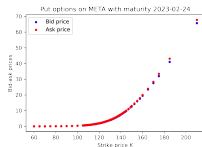


$$S_0^1 = \alpha \underline{S}_0^1 + (1 - \alpha) \bar{S}_0^1 \leftarrow \text{--- } (\underline{S}_0^1, \bar{S}_0^1) \quad S_1^1$$

$$C_1^K = \max\{S_1^1 - K\}$$



$$P_1^K = \max\{K - S_1^1\}$$

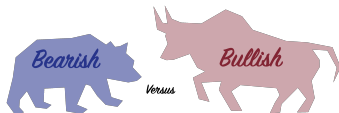
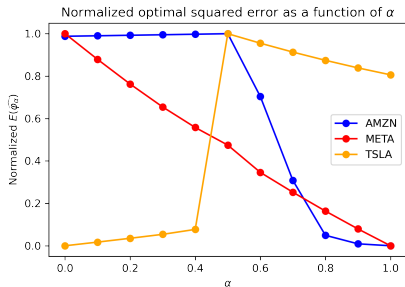


Tuning of α : a measure of market pessimism

Goal

For a fixed $\alpha \in [0, 1]$, look for $\widehat{\varphi}_\alpha$ minimizing

$$E(\widehat{\varphi}_\alpha) = \sum_{K \in \mathcal{K}_{call}} \left(C_0^{K,\alpha} - \frac{C_{\widehat{\varphi}_\alpha}[C_1^K]}{1+r} \right)^2 + \sum_{K \in \mathcal{K}_{put}} \left(P_0^{K,\alpha} - \frac{C_{\widehat{\varphi}_\alpha}[P_1^K]}{1+r} \right)^2$$



Thanks for your attention!



Our poster

No-Arbitrage Pricing with α -DS Mixtures in a Market with Bid-Ask Spreads


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Motivation: pricing in a market with frictions

A classical no-arbitrage pricing theory assumes that the market's underlying asset follows a Black-Scholes model with a constant volatility. However, this is not supported by empirical evidence. An alternative is to consider an α -DS mixture model.

PROPOSITION: Market price bubbles, existing in the case of bid-ask spreads.

THEOREM: Market price bubbles, existing in the case of bid-ask spreads.



No-arbitrage pricing under α -PRU

Given a payoff ϕ at time T , $\phi = \phi^+ - \phi^-$, $\phi^+ \in \mathcal{C}^2$ and $\phi^- \in \mathcal{C}^1$.

Price at time $t = 0$: $V_0 = \mathbb{E}[\phi]$

Payoff under α -PRU at time $t = 0$: $V_0 = \mathbb{E}[\phi]$

α -PRU principle at time $T = 1$

PRU (Partial Hedging Underwriting): the agent may only require that $R \geq R_{PRU}$ success, without hedging underwriting costs.

Theorem (First FTAP under α -PRU):

Let $\phi \in \mathcal{C}^1$. The following conditions are equivalent:

(i) $V_0 = \mathbb{E}[\phi]$ is the unique no-arbitrage price in a market with bid-ask spreads.

(ii) $V_0 = \mathbb{E}[\phi]$ is the unique no-arbitrage price in a market with bid-ask spreads.

Theorem (Second FTAP under α -PRU):

Let $\phi \in \mathcal{C}^1$. The unique no-arbitrage price in a market with bid-ask spreads and α -PRU is $V_0 = \mathbb{E}[\phi]$.

α -DS mixture no-arbitrage price of a payoff ϕ at $T = 1$

$$V_0 = \mathbb{E}[\phi] = \int_{-\infty}^{\infty} \phi(x) f_{\alpha}(x) dx$$

α -DS mixtures

Definition: Let $\alpha > 0$. A function f is called an α -DS mixture if there exists a bid function $B(x)$ and an ask function $A(x)$ such that $f(x) = \int_{-\infty}^{\infty} \delta(x-y) f_{\alpha}(y) dy$.

Definition (α -DS mixture):

Let $\alpha > 0$. A function f is called an α -DS mixture if there exists a bid function $B(x)$ and an ask function $A(x)$ such that $f(x) = \int_{-\infty}^{\infty} \delta(x-y) f_{\alpha}(y) dy$.

The bid function $B(x)$ is used to represent the α -DS mixture f .

We further characterize the existence of bid and ask functions in DS mixtures.

Proposition (unique representation):

Let $f \in \mathcal{C}^1$. Then f is an α -DS mixture if and only if there exists a bid function $B(x)$ and an ask function $A(x)$ such that $f(x) = \int_{-\infty}^{\infty} \delta(x-y) f_{\alpha}(y) dy$.

Properties of α -DS mixtures

Proposition (properties of f_{α}):

Let $\alpha > 0$. Then f_{α} is an α -DS mixture. $f_{\alpha}(x) > 0$. $f_{\alpha}(x)$ satisfies the following properties:

(i) $f_{\alpha}(x) > 0$ for all $x \in \mathbb{R}$.

(ii) $f_{\alpha}(x) > 0$ for all $x \in \mathbb{R}$.

(iii) $f_{\alpha}(x) > 0$ for all $x \in \mathbb{R}$.

(iv) $f_{\alpha}(x) > 0$ for all $x \in \mathbb{R}$.

For every $\alpha > 0$, the class \mathcal{M}_{α} of all α -DS mixtures is \mathcal{M}_{α} is convex and contains the class of all probability measures in \mathcal{M}_{α} .

α -DS mixture Choquet expectation

Given a payoff ϕ at time T , $\phi = \phi^+ - \phi^-$, $\phi^+ \in \mathcal{C}^2$ and $\phi^- \in \mathcal{C}^1$.

Choquet expectation $\mathbb{E}^{\alpha}[\phi] = \int_{-\infty}^{\infty} \phi(x) f_{\alpha}(x) dx$.

Proposition: The Choquet expectation $\mathbb{E}^{\alpha}[\phi]$ is the unique no-arbitrage price in a market with bid-ask spreads.



RETA stock market data with bid-ask spreads

Consider a single risky asset.

$\alpha = 1$ (bid-ask spread) $\alpha = 1$ (bid-ask spread)

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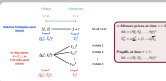
One-period market with bid-ask spreads

Consider a one-period market with bid-ask spreads.

Asset price at time $t = 0$: $V_0 = \mathbb{E}[\phi]$

Asset price at time $t = 1$: $V_1 = \mathbb{E}[\phi]$

Payoff at time $t = 1$: $\phi = \phi^+ - \phi^-$



Tuning of α : a measure of market pessimism

For a fixed $\alpha > 0$, consider the no-arbitrage price $V_0 = \mathbb{E}[\phi]$ and $V_1 = \mathbb{E}[\phi]$.

$V_0 = \mathbb{E}[\phi]$

$V_1 = \mathbb{E}[\phi]$

$V_0 = \mathbb{E}[\phi]$

$V_1 = \mathbb{E}[\phi]$

