No-Arbitrage Pricing with α-DS Mixtures in a Market with Bid-Ask Spreads

[a]Davide Petturiti [b]Barbara Vantaggi

[a]Dept. Economics, University of Perugia, Italy
[b]Dept. MEMOTEF, Sapienza University of Rome, Italy

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Silvia Marconi

- NRRP project: "Future Artificial Intelligence Research"
- PRIN project: "Models for dynamic reasoning under partial knowledge to make interpretable decisions"

Classical "ideal" financial market

PERFECT MARKET HYPOTHESES:

Absence of frictions:

- No taxes nor transaction costs;
- Infinitely divisible securities;
- Unlimited short sale;

$\Longrightarrow$ Selling and buying prices of securities coincide

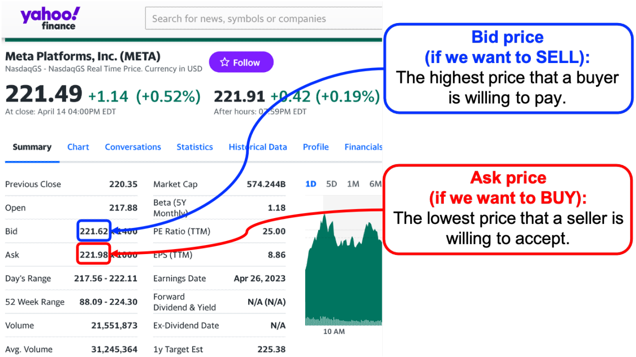
Competitiveness:

- Profit maximizer agents;
- Price-taker agents.

Classical no-arbitrage principle

Prices can be expressed as **discounted expectations** with respect to an "artificial" probability measure Q.

Is the absence of frictions hypothesis realistic?



OUR GOAL: Replace Q with a non-additive measure so as to consider bid-ask spreads.

α-DS mixtures

Consider:

- $\Omega = \{1, \ldots, n\}$ with $n \geq 1$, a finite set of states of the world
- $\mathcal{P}(\Omega)$, power set of events

Definition

Let $\alpha \in [0,1]$. A mapping $\varphi_\alpha : \mathcal{P}(\Omega) \to [0,1]$ is called an α-**DS mixture** if there exists a belief function $Bel : \mathcal{P}(\Omega) \to [0,1]$ with dual plausibility function Pl such that, for all $A \in \mathcal{P}(\Omega)$,

$$\begin{aligned} \varphi_\alpha(A) &= \alpha Bel(A) + (1-\alpha) Pl(A) \\ &= \alpha Bel(A) + (1-\alpha)(1 - Bel(A^c)). \end{aligned}$$

The belief function Bel is said to **represent** the α-DS mixture φ_α.

$\Longrightarrow$ The representation is unique when $\alpha \neq \frac{1}{2}$

α-DS mixture Choquet expectation

Every φ_α uniquely extends to a functional $\mathbb{C}_{\varphi_\alpha} : \mathbb{R}^\Omega \to \mathbb{R}$ by setting, for every $X \in \mathbb{R}^\Omega$,

$$\mathbb{C}_{\varphi_\alpha}[X] = \oint X \, \mathrm{d}\varphi_\alpha$$

Hurwicz-like representation:

$$\mathbb{C}_{\varphi_\alpha}[X] = \alpha \min_{P \in \mathcal{C}_{Bel}} \mathbb{E}_P[X] + (1-\alpha) \max_{P \in \mathcal{C}_{Bel}} \mathbb{E}_P[X]$$

where $\mathcal{C}_{Bel}$ is the core of Bel

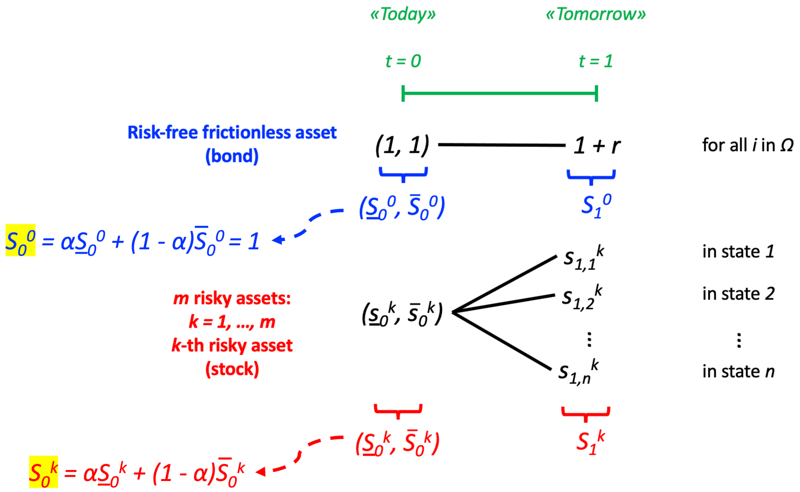
Möbius-like representation:

$$\mathbb{C}_{\varphi_\alpha}[X] = \sum_{B \in \mathcal{U}} [\![X]\!]^\alpha(B)\mu(B)$$

where μ is the Möbius inverse of Bel and

$$\mathcal{U} = \mathcal{P}(\Omega) \setminus \{\emptyset\} \text{ and } [\![X]\!]^\alpha(B) = \alpha \min_{i \in B} X(i) + (1-\alpha) \max_{i \in B} X(i)$$

One-period market with bid-ask spreads



No-arbitrage pricing under α-PRU

Given a portfolio $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_m)^T \in \mathbb{R}^{m+1}$ we define:

Price at time $t = 0$: $V_0^{\boldsymbol{\lambda}} = \lambda_0 + \sum_{k=1}^{m} \lambda_k S_0^k$

Payoff under α-PRU at time $t = 1$: $V_1^{\boldsymbol{\lambda}} = \lambda_0(1 + r) + \sum_{k=1}^{m} \lambda_k [\![S_1^k]\!]^\alpha$

α-PRU principle at time $t = 1$

PRU (Partially Resolving Uncertainty) due to Jaffray: An agent may only acquire that $B \neq \emptyset$ occurs, without knowing which is the true $i \in B$

α-pessimism: An agent always considers the α-mixture between the minimum and the maximum of random payoffs on every $B \neq \emptyset$

First and Second FTAP under α-PRU

Theorem (First FTAP under α-PRU)

Let $\alpha \in [0,1]$. The following conditions are equivalent:

(i) there exists an α-DS mixture $\widehat{\varphi_\alpha}$ represented by a belief function strictly positive on $\mathcal{U}$ and such that $\frac{\mathbb{C}_{\widehat{\varphi_\alpha}}[S_1^k]}{1+r} = S_0^k$, for $k = 1, \ldots, m$;

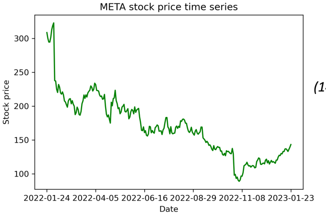
(ii) for every $\boldsymbol{\lambda} \in \mathbb{R}^{m+1}$ none of the following conditions holds:

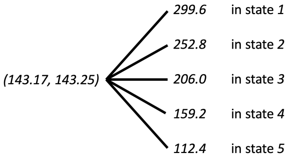
(a) $\underbrace{V_0^{\boldsymbol{\lambda}} < 0}_{\text{we are paid today}}$ and $\underbrace{V_1^{\boldsymbol{\lambda}} \geq 0 \text{ and } V_1^{\boldsymbol{\lambda}}(\{i\}) = 0, \text{ for all } i}_{\text{in all events we do not lose money and have a null gain on states tomorrow}}$;

(b) $\underbrace{V_0^{\boldsymbol{\lambda}} \leq 0}_{\text{we do not pay today}}$ and $\underbrace{V_1^{\boldsymbol{\lambda}} \geq 0 \text{ with } V_1^{\boldsymbol{\lambda}}(\{i\}) > 0 \text{ for some } i}_{\text{in all events we do not lose money and have a positive gain in a state tomorrow}}$.

$\Longrightarrow$ The Second FTAP under α-PRU gives a sufficient condition (α-**PRU completeness**) for uniqueness of $\widehat{\varphi_\alpha}$

META stock market data with bid-ask spreads

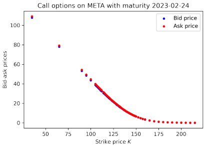


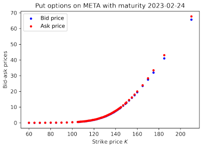


$(\underline{S}_0^1, \overline{S}_0^1)$ $\quad S_1^1$

$S_0^1 = \alpha \underline{S}_0^1 + (1 - \alpha)\overline{S}_0^1$

$$C_1^K = \max\{S_1^1 - K\} \qquad P_1^K = \max\{K - S_1^1\}$$



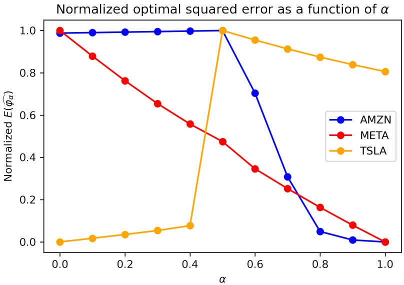


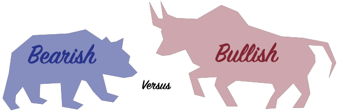
Tuning of α: a measure of market pessimism

Goal

For a fixed $\alpha \in [0,1]$, look for $\widehat{\varphi_\alpha}$ minimizing

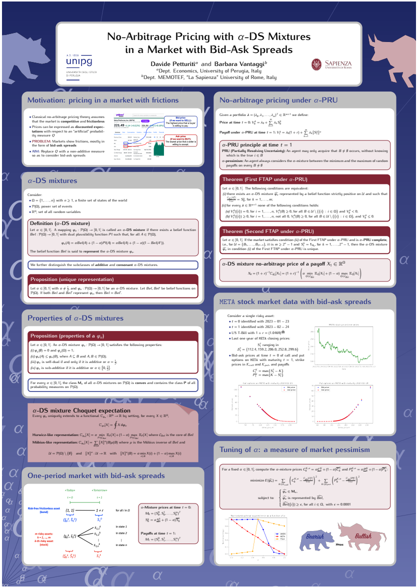
$$E(\widehat{\varphi_\alpha}) = \sum_{K \in \mathcal{K}_{call}} \left(C_0^{K,\alpha} - \frac{\mathbb{C}_{\widehat{\varphi_\alpha}}[C_1^K]}{1+r} \right)^2 + \sum_{K \in \mathcal{K}_{put}} \left(P_0^{K,\alpha} - \frac{\mathbb{C}_{\widehat{\varphi_\alpha}}[P_1^K]}{1+r} \right)^2$$





Thanks for your attention!

Our poster



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Motivation: pricing in a market with frictions
α-DS mixtures
Definition (α-DS mixture)
Proposition (unique representation)
Properties of α-DS mixtures
Proposition (properties of a φα)
α-DS mixture Choquet expectation
One-period market with bid-ask spreads
No-arbitrage pricing under α-PRU
α-PRU principle at time t = 1
Theorem (First FTAP under α-PRU)
Theorem (Second FTAP under α-PRU)
META stock market data with bid-ask spreads
Tuning of α: a measure of market pessimism
Bearish
Bullish