Desirable sets of things and their logic

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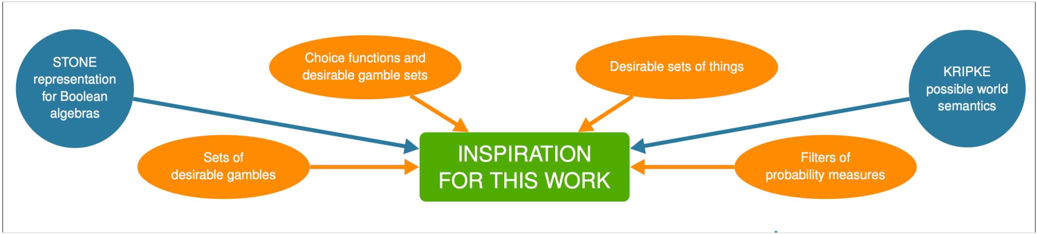
STONE representation for Boolean algebras
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Desirable sets of things
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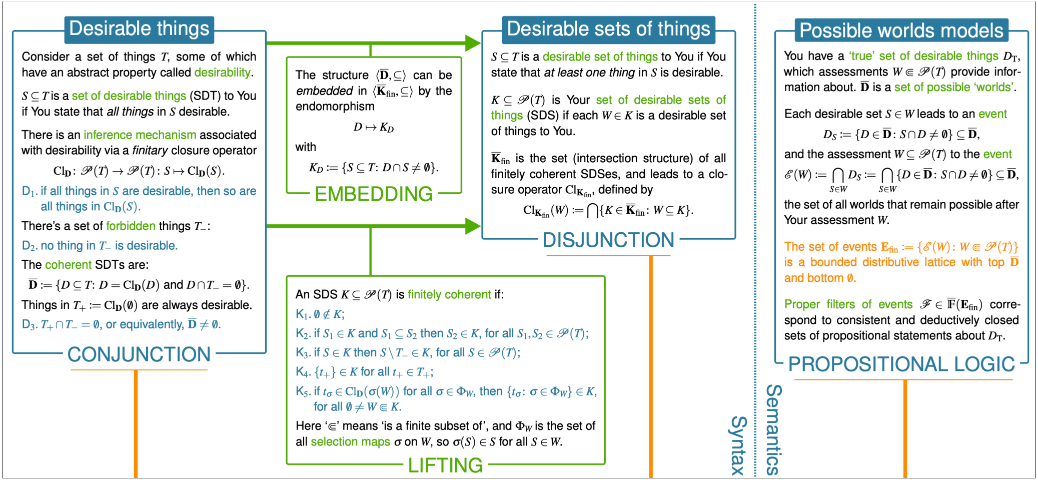
INSPIRATION FOR THIS WORK
Filters of probability measures


Desirable things
Consider a set of things T, some of which have an abstract property called desirability.
$S \subseteq T$ is a set of desirable things (SDT) to You if You state that *all things* in S desirable.
There is an inference mechanism associated with desirability via a *finitary* closure operator
$\mathrm{Cl}_{\mathbf{D}} \colon \mathscr{P}(T) \to \mathscr{P}(T) \colon S \mapsto \mathrm{Cl}_{\mathbf{D}}(S)$.
D$_1$. if all things in S are desirable, then so are all things in $\mathrm{Cl}_{\mathbf{D}}(S)$.
There's a set of forbidden things T_-:
D$_2$. no thing in T_- is desirable.
The coherent SDTs are:
$\overline{\mathbf{D}} := \{D \subseteq T \colon D = \mathrm{Cl}_{\mathbf{D}}(D)$ and $D \cap T_- = \emptyset\}$.
Things in $T_+ := \mathrm{Cl}_{\mathbf{D}}(\emptyset)$ are always desirable.
D$_3$. $T_+ \cap T_- = \emptyset$, or equivalently, $\overline{\mathbf{D}} \neq \emptyset$.
CONJUNCTION
The structure $\langle \overline{\mathbf{D}}, \subseteq \rangle$ can be *embedded* in $\langle \overline{\mathbf{K}}_{\mathrm{fin}}, \subseteq \rangle$ by the endomorphism
$D \mapsto K_D$
with
$K_D := \{S \subseteq T \colon D \cap S \neq \emptyset\}$.
EMBEDDING
Desirable sets of things
$S \subseteq T$ is a desirable set of things to You if You state that *at least one thing* in S is desirable.
$K \subseteq \mathscr{P}(T)$ is Your set of desirable sets of things (SDS) if each $W \in K$ is a desirable set of things to You.
$\overline{\mathbf{K}}_{\mathrm{fin}}$ is the set (intersection structure) of all finitely coherent SDSes, and leads to a closure operator $\mathrm{Cl}_{\overline{\mathbf{K}}_{\mathrm{fin}}}$, defined by
$\mathrm{Cl}_{\overline{\mathbf{K}}_{\mathrm{fin}}}(W) := \bigcap \{K \in \overline{\mathbf{K}}_{\mathrm{fin}} \colon W \subseteq K\}$.
DISJUNCTION
An SDS $K \subseteq \mathscr{P}(T)$ is finitely coherent if:
K$_1$. $\emptyset \notin K$;
K$_2$. if $S_1 \in K$ and $S_1 \subseteq S_2$ then $S_2 \in K$, for all $S_1, S_2 \in \mathscr{P}(T)$;
K$_3$. if $S \in K$ then $S \setminus T_- \in K$, for all $S \in \mathscr{P}(T)$;
K$_4$. $\{t_+\} \in K$ for all $t_+ \in T_+$;
K$_5$. if $t_\sigma \in \mathrm{Cl}_{\mathbf{D}}(\sigma(W))$ for all $\sigma \in \Phi_W$, then $\{t_\sigma \colon \sigma \in \Phi_W\} \in K$, for all $\emptyset \neq W \Subset K$.
Here '$\Subset$' means 'is a finite subset of', and Φ_W is the set of all selection maps σ on W, so $\sigma(S) \in S$ for all $S \in W$.
LIFTING
Syntax
Semantics
Possible worlds models
You have a 'true' set of desirable things D_{T}, which assessments $W \Subset \mathscr{P}(T)$ provide information about. $\overline{\mathbf{D}}$ is a set of possible 'worlds'.
Each desirable set $S \in W$ leads to an event
$D_S := \{D \in \overline{\mathbf{D}} \colon S \cap D \neq \emptyset\} \subseteq \overline{\mathbf{D}}$,
and the assessment $W \subseteq \mathscr{P}(T)$ to the event
$\mathscr{E}(W) := \bigcap_{S \in W} D_S := \bigcap_{S \in W} \{D \in \overline{\mathbf{D}} \colon S \cap D \neq \emptyset\} \subseteq \overline{\mathbf{D}}$,
the set of all worlds that remain possible after Your assessment W.
The set of events $\mathbf{E}_{\mathrm{fin}} := \{\mathscr{E}(W) \colon W \Subset \mathscr{P}(T)\}$ is a bounded distributive lattice with top $\overline{\mathbf{D}}$ and bottom $\emptyset$.
Proper filters of events $\mathscr{F} \in \overline{\mathbb{F}}(\mathbf{E}_{\mathrm{fin}})$ correspond to consistent and deductively closed sets of propositional statements about D_{T}.
PROPOSITIONAL LOGIC

Complete SDSes

A finitely coherent SDS $K \in \overline{\mathbf{K}}_{\mathrm{fin}}$ is complete if

C. $(\forall S_1, S_2 \subseteq T)(S_1 \cup S_2 \in K \Rightarrow (S_1 \in K \text{ or } S_2 \in K))$.

$\overline{\mathbf{K}}_{\mathrm{fin,c}}$ is the set of all complete and finitely coherent SDSes.

The established order isomorphism allows us to translate the Prime Filter Representation Theorem into:

An SDS K is finitely coherent if and only if it is the *non-empty* intersection of all the complete and finitely coherent SDSes it is included in:

$$K = \bigcap \underbrace{\{K' \in \overline{\mathbf{K}}_{\mathrm{fin,c}} : K \subseteq K'\}}_{\neq \emptyset}.$$

REPRESENTATION

ORDER ISOMORPHISM

The structures $\langle \overline{\mathbf{K}}_{\mathrm{fin}}, \subseteq \rangle$ and $\langle \overline{\mathbb{F}}(\mathbf{E}_{\mathrm{fin}}), \subseteq \rangle$ are order isomorphic, via the order isomorphisms

$$\varphi^{\mathrm{fin}}_{\mathbf{D}}(K) := \{\mathscr{E}(W) : W \in K\},$$

and

$$\kappa^{\mathrm{fin}}_{\mathbf{D}}(\mathscr{F}) := \{S \subseteq T : \overline{\mathbf{D}}_S \in \mathscr{F}\}.$$

Prime filters

A proper filter $\mathscr{F} \in \overline{\mathbb{F}}(\mathbf{E}_{\mathrm{fin}})$ is prime if

PF. $(\forall E_1, E_2 \in \mathbf{E}_{\mathrm{fin}})(E_1 \cup E_2 \in \mathscr{F} \Rightarrow (E_1 \in \mathscr{F} \text{ or } E_2 \in \mathscr{F}))$.

$\overline{\mathbb{F}}_{\mathrm{p}}(\mathbf{E}_{\mathrm{fin}})$ is the set of all prime filters.

The well-known Prime Filter Representation Theorem states that:

A set of events $\mathscr{F}$ is a proper filter if and only if it is the *non-empty* intersection of all the prime filters it is included in:

$$\mathscr{F} = \bigcap \underbrace{\{\mathscr{G} \in \overline{\mathbb{F}}_{\mathrm{p}}(\mathbf{E}_{\mathrm{fin}}) : \mathscr{F} \subseteq \mathscr{G}\}}_{\neq \emptyset}.$$

REPRESENTATION

Finitary SDSes

We concentrate on the *finite sets of things* in

$$\mathcal{Q}(T) := \{S \in \mathcal{P}(T) : S \Subset T\}.$$

For any SDS $W \subseteq \mathcal{P}(T)$, we call

$$\mathrm{fin}(W) := W \cap \mathcal{Q}(T)$$

its finite part, and collect all its sets with finite desirable subsets in

$$\mathrm{fty}(W) := \{S \in \mathcal{P}(T) : (\exists \hat{S} \in W \cap \mathcal{Q}(T))\hat{S} \Subset S\},$$

its finitary part.

An SDS $W \subseteq \mathcal{P}(T)$ is called finitary if all its desirable sets have finite desirable subsets, so

$$W \subseteq \mathrm{fty}(W).$$

A finitely coherent SDS K is finitary iff $K = \mathrm{fty}(K)$.

Conjunctive SDSes

A conjunctive SDS $W \subseteq \mathcal{P}(T)$ is a finitary SDS all of whose minimal elements are singletons:

$$(\forall S \in W)(\exists t \in S)\{t\} \in W.$$

A finitely coherent SDS K is conjunctive if and only if there is some coherent SDT D such that $K = K_D$ and then necessarily:

$$D = \{t \in T : \{t\} \in K\}.$$

The finitary part of any finitely coherent and complete SDS is finitely coherent and conjunctive; consequently, any finitary and finitely coherent SDS is complete if and only if it is conjunctive.

Note: the paper also discusses and studies stronger, infinitary versions of the lifting axioms K1–K5.

REPRESENTATION

A *finitary* SDS K is finitely coherent if and only if it is the *non-empty* intersection of all the finitely coherent conjunctive SDSes it is included in:

$$K = \bigcap \underbrace{\{K_D : D \in \overline{\mathbf{D}} \text{ and } K \subseteq K_D\}}_{\neq \emptyset}.$$

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