Sets of probability measures and convex combination spaces

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Sets of probability measures

The L^1 **Wasserstein metric** in $W_1(\mathbb{R})$, the space of probability measures with finite expectation is defined as $w_1(P,Q) = \inf_{\mathcal{L}(X)=P,\mathcal{L}(Y)=Q} E[|X-Y|]$.

Its generalization to sets of probabilities is denoted by $\mathcal{W}_1(\mathcal{P},\mathcal{Q})$ and agrees with the Hausdorff metric.

Denote by $\mathcal{P}_1(\mathbb{R})$ the set of all sets $\mathcal{P}$ of probability measures such that

- $\mathcal{P}$ is weakly compact,
- For an arbitrary point $r \in \mathbb{R}$, $\lim_{k\to\infty} \mathbb{E}^{\mathcal{P}}[d(\cdot,r) I_{\{x\in\mathbb{R}: d(r,x)\geq K\}}]=0$.

Denote by $\mathcal{P}_1^c(\mathbb{R})$ the subset of $\mathcal{P}_1(\mathbb{R})$ which contains only the convex sets of probabilities.

Convex combination spaces

Let ($\mathbb{E}$,d) be a metric space with a convex combination operation which for $\lambda_1,\ldots,\lambda_n$ with $\sum_{i=1}^n \lambda_i=1$ and any $v_1,\ldots,v_n \in \mathbb{E}$ produces an element of $\mathbb{E}$ denoted $[\lambda_i, v_i]_{i=1}^n$

Assume that it satisfies:

- (CC1) Commutativity,
- (CC2) Associativity,
- (CC3) Continuity,
- (CC4) Negative curvature,
- (CC5) Convexification, that is, for each $v \in \mathbb{E}$, there exists $\lim_{n\to\infty}[n^{-1}, v]_{i=1}^n$, which will be denoted by $\mathbf{K}_{\mathbb{E}}(v)$

Then ($\mathbb{E}$,d) is a **convex combination space**.

Convex combinations based on convolution

- $W_1(\mathbb{R})$ is a convex combination space when endowed with the Wasserstein metric.
- $\mathcal{P}_1(\mathbb{R}) = \mathcal{K}(W_1(\mathbb{R}))$.
- $\mathcal{P}_1(\mathbb{R})$ is a convex combination space when endowed with the generalized Wasserstein metric.
- $\mathcal{P}_1^c(\mathbb{R})$ is a convex combination space as a subset of $(\mathcal{P}_1(\mathbb{R}), \mathcal{W}_1)$.

Applications

- Let $P \in W_1(\mathbb{R})$. Then $\{P\}$ is convex if and only if P is a degenerate distribution.
- Let $P \in \mathcal{P}_1(\mathbb{R})$. Then $\mathbf{K}_{\mathcal{P}_1(\mathbb{R})}(\mathcal{P}) = \{\delta_x : x \in [\underline{b}(\mathcal{P}), \overline{b}(\mathcal{P})]\}$.
- Let $P \in \mathcal{P}_1(\mathbb{R})$. Then $d(\mathcal{L}(n^{-1}\sum_{i=1}^n X_i), \{\delta_x : x \in [\underline{b}(\mathcal{P}), \overline{b}(\mathcal{P})]\}) \to 0$ uniformly over all independent sequences $\{X_n\}$ such that $\mathcal{L}(X_n) \in \mathcal{P}$ for all $n \in \mathbb{N}$.
- **Strong law of large numbers:** Let Γ be an integrable random element of $\mathcal{P}_1(\mathbb{R})$. Let $\{\Gamma_n\}$ be pairwise independent random elements of $\mathcal{P}_1(\mathbb{R})$ identically distributed as Γ. Then $\mathcal{W}_1([n^{-1}, \Gamma_i]_{i=1}^n, E(\Gamma)) \to 0$ almost surely.
- **Jensen's inequality:** Let $\varphi: \mathcal{P}_1(\mathbb{R}) \to \mathbb{R}$ be a lower semicontinuous function and midpoint convex. Let Γ be an integrable random element of $\mathcal{P}_1(\mathbb{R})$ such that $E(\varphi(\Gamma)) < \infty$. Then $\varphi(E(\Gamma)) \leq E(\varphi(\Gamma))$.
- **Dominated convergence theorem:** Let Γ_n, Γ be random elements of $\mathcal{P}_1(\mathbb{R})$ such that $\mathcal{W}_1(\Gamma_n, \{\delta_0\}) \leq g$ for some $g \in L^1(\Omega, \mathcal{A}, P)$. If $\Gamma_n \to \Gamma$ weakly then $\mathcal{W}_1(E(\Gamma_n), E(\Gamma)) \to 0$.

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