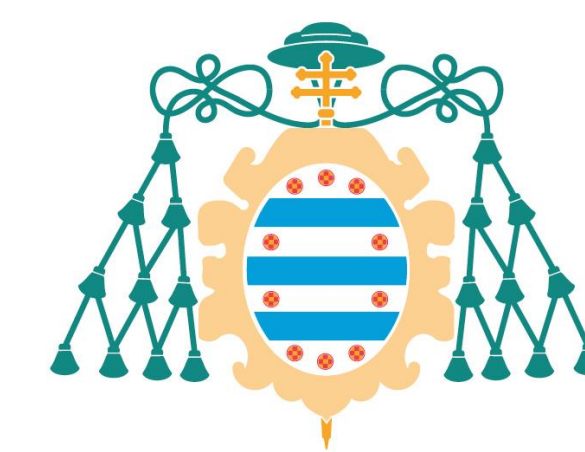


# Sets of probability measures and convex combination spaces

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## Sets of probability measures

The  $L^1$  **Wasserstein metric** in  $W_1(\mathbb{R})$ , the space of probability measures with finite expectation is defined as  $w_1(P, Q) = \inf_{\mathcal{L}(X)=P, \mathcal{L}(Y)=Q} E[|X-Y|]$ .

Its generalization to sets of probabilities is denoted by  $\mathcal{W}_1(\mathcal{P}, \mathcal{Q})$  and agrees with the Hausdorff metric.

Denote by  $\mathcal{P}_1(\mathbb{R})$  the set of all sets  $\mathcal{P}$  of probability measures such that

- $\mathcal{P}$  is weakly compact,
- For an arbitrary point  $r \in \mathbb{R}$ ,  $\lim_{k \rightarrow \infty} E^{\mathcal{P}}[d(\cdot, r) I_{\{x \in \mathbb{R}: d(r, x) \geq k\}}] = 0$ .

Denote by  $\mathcal{P}_1^c(\mathbb{R})$  the subset of  $\mathcal{P}_1(\mathbb{R})$  which contains only the convex sets of probabilities.

## Convex combination spaces

Let  $(\mathbb{E}, d)$  be a metric space with a convex combination operation which for  $\lambda_1, \dots, \lambda_n$  with  $\sum_{i=1}^n \lambda_i = 1$  and any  $v_1, \dots, v_n \in \mathbb{E}$  produces an element of  $\mathbb{E}$  denoted  $[\lambda_i, v_i]_{i=1}^n$

Assume that it satisfies:

- (CC1) Commutativity,
- (CC2) Associativity,
- (CC3) Continuity,
- (CC4) Negative curvature,
- (CC5) Convexification, that is, for each  $v \in \mathbb{E}$ , there exists

$\lim_{n \rightarrow \infty} [n^{-1}, v]_{i=1}^n$ , which will be denoted by  $\mathbf{K}_{\mathbb{E}}(v)$

Then  $(\mathbb{E}, d)$  is a **convex combination space**.

## Convex combinations based on convolution

- $W_1(\mathbb{R})$  is a convex combination space when endowed with the Wasserstein metric.
- $\mathcal{P}_1(\mathbb{R}) = \mathcal{K}(W_1(\mathbb{R}))$ .
- $\mathcal{P}_1(\mathbb{R})$  is a convex combination space when endowed with the generalized Wasserstein metric.
- $\mathcal{P}_1^c(\mathbb{R})$  is a convex combination space as a subset of  $(\mathcal{P}_1(\mathbb{R}), \mathcal{W}_1)$ .

## Applications

- Let  $P \in W_1(\mathbb{R})$ . Then  $\{P\}$  is convex if and only if  $P$  is a degenerate distribution.
- Let  $P \in \mathcal{P}_1(\mathbb{R})$ . Then  $\mathbf{K}_{\mathcal{P}_1(\mathbb{R})}(P) = \{\delta_x: x \in [\underline{b}(P), \bar{b}(P)]\}$ .
- Let  $P \in \mathcal{P}_1(\mathbb{R})$ . Then  $d(\mathcal{L}(n^{-1} \sum_{i=1}^n X_i), \{\delta_x: x \in [\underline{b}(P), \bar{b}(P)]\}) \rightarrow 0$  uniformly over all independent sequences  $\{X_n\}$  such that  $\mathcal{L}(X_n) \in \mathcal{P}$  for all  $n \in \mathbb{N}$ .
- **Strong law of large numbers:** Let  $\Gamma$  be an integrable random element of  $\mathcal{P}_1(\mathbb{R})$ . Let  $\{\Gamma_n\}$  be pairwise independent random elements of  $\mathcal{P}_1(\mathbb{R})$  identically distributed as  $\Gamma$ . Then  $\mathcal{W}_1([n^{-1}, \Gamma_i]_{i=1}^n, E(\Gamma)) \rightarrow 0$  almost surely.
- **Jensen's inequality:** Let  $\varphi: \mathcal{P}_1(\mathbb{R}) \rightarrow \mathbb{R}$  be a lower semicontinuous function and midpoint convex. Let  $\Gamma$  be an integrable random element of  $\mathcal{P}_1(\mathbb{R})$  such that  $E(\varphi(\Gamma)) < \infty$ . Then  $\varphi(E(\Gamma)) \leq E(\varphi(\Gamma))$ .
- **Dominated convergence theorem:** Let  $\Gamma_n, \Gamma$  be random elements of  $\mathcal{P}_1(\mathbb{R})$  such that  $\mathcal{W}_1(\Gamma_n, \{\delta_0\}) \leq g$  for some  $g \in L^1(\Omega, \mathcal{A}, P)$ . If  $\Gamma_n \rightarrow \Gamma$  weakly then  $\mathcal{W}_1(E(\Gamma_n), E(\Gamma)) \rightarrow 0$ .

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