Measuring Deviation from Stochastic Dominance under Imprecision

Juan Baz[1], **Ander Gray**[2] and **Raúl Pérez-Fernández**[1]

[1]Department of Statistics and Operations Research and Mathematics Didactics, University of Oviedo, Spain
[2]Culham Science Centre, UKAEA, UK
bazjuan@uniovi.es, ander.gray@ukaea.uk, perezfernandez@uniovi.es

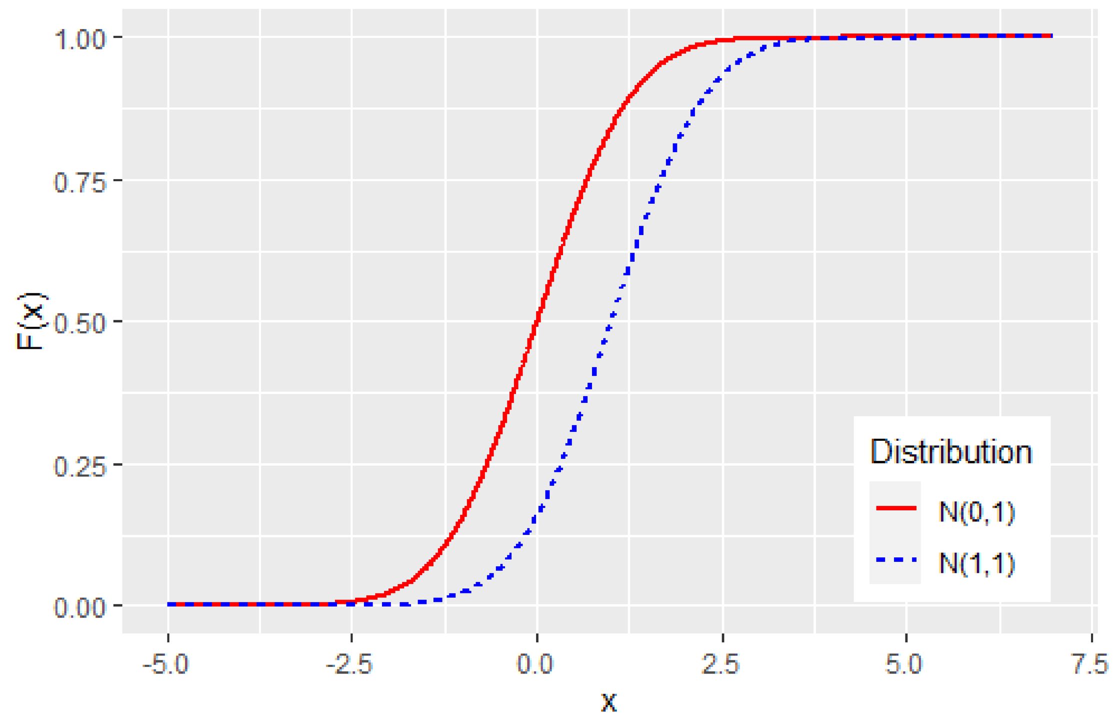
1. Introduction to stochastic orders and stochastic dominance

Stochastic orders [2, 5] have been successfully used in probability and statistics and in many related fields of application for the comparison of random variables.

The most classical stochastic order is stochastic dominance. A random variable X stochastically dominates another random variable Y, denoted by $X \succeq_{\text{FSD}} Y$, if

$$F_X(x) \leq F_Y(x), \quad \forall x \in \mathbb{R}.$$

For instance, let $X \rightsquigarrow N(1,1)$ and $Y \rightsquigarrow N(0,1)$ be two normal distributions. It holds that $X \succeq_{\text{FSD}} Y$, since F_X is pointwisely smaller than F_y as illustrated in the following figure:



An interesting property of stochastic dominance is that:

$$X \succeq_{\text{FSD}} Y \quad \Rightarrow \quad E(X) \geq E(Y).$$

The converse implication is true within a location distribution family.

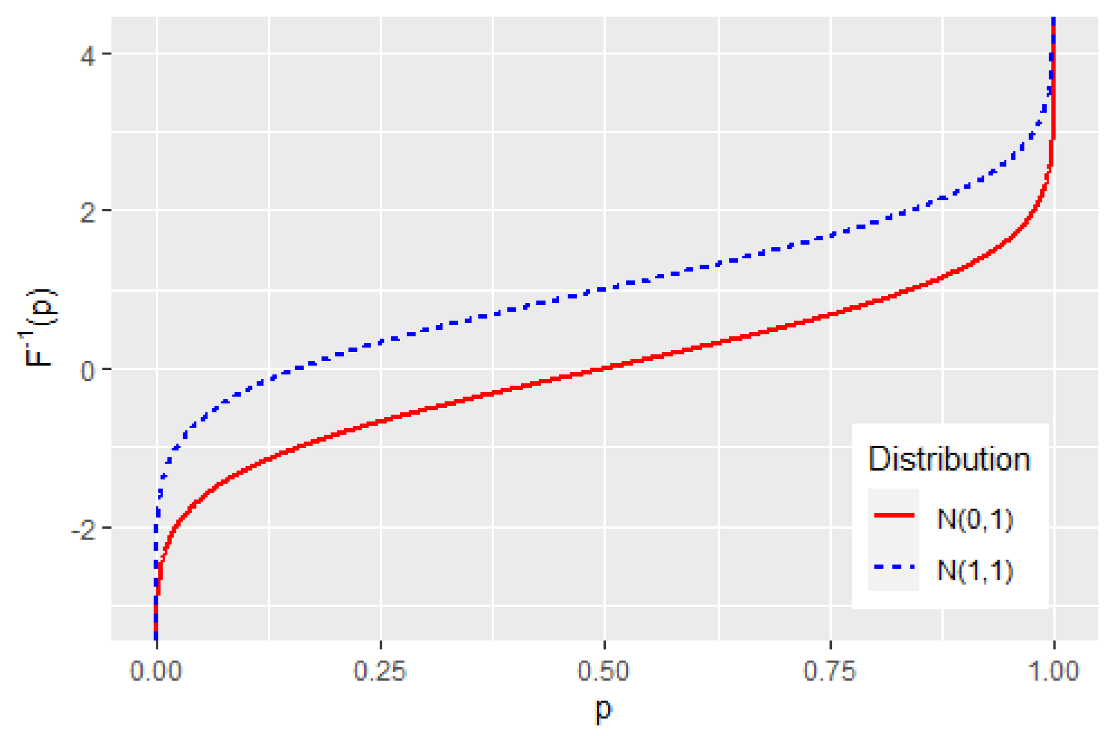
2. The γ-index

Unfortunately, stochastic dominance does not hold in practice. For instance, within a location-scale family generated from a standard distribution with unbounded support (such as the normal distribution family), there do not exist two random variables with different variance such that one of the random variables stochastically dominates the other. Still, if the location parameters of both random variables are very far from one another, this stochastic dominance would be 'very close' to being satisfied.

It is for such purpose that a measure of deviation from stochastic dominance is of interest. For the proposal of such a measure, we could resort to a well-known property of stochastic dominance, which may also be expressed in terms of the quantile functions:

$$X \succeq_{\text{FSD}} Y \quad \Leftrightarrow \quad F_X^{-1}(p) \geq F_Y^{-1}(p), \quad \forall p \in]0,1[\,.$$

For $X \rightsquigarrow N(1,1)$ and $Y \rightsquigarrow N(0,1)$, it can be seen that F_X^{-1} is pointwisely greater than F_y^{-1} as illustrated in the following figure:

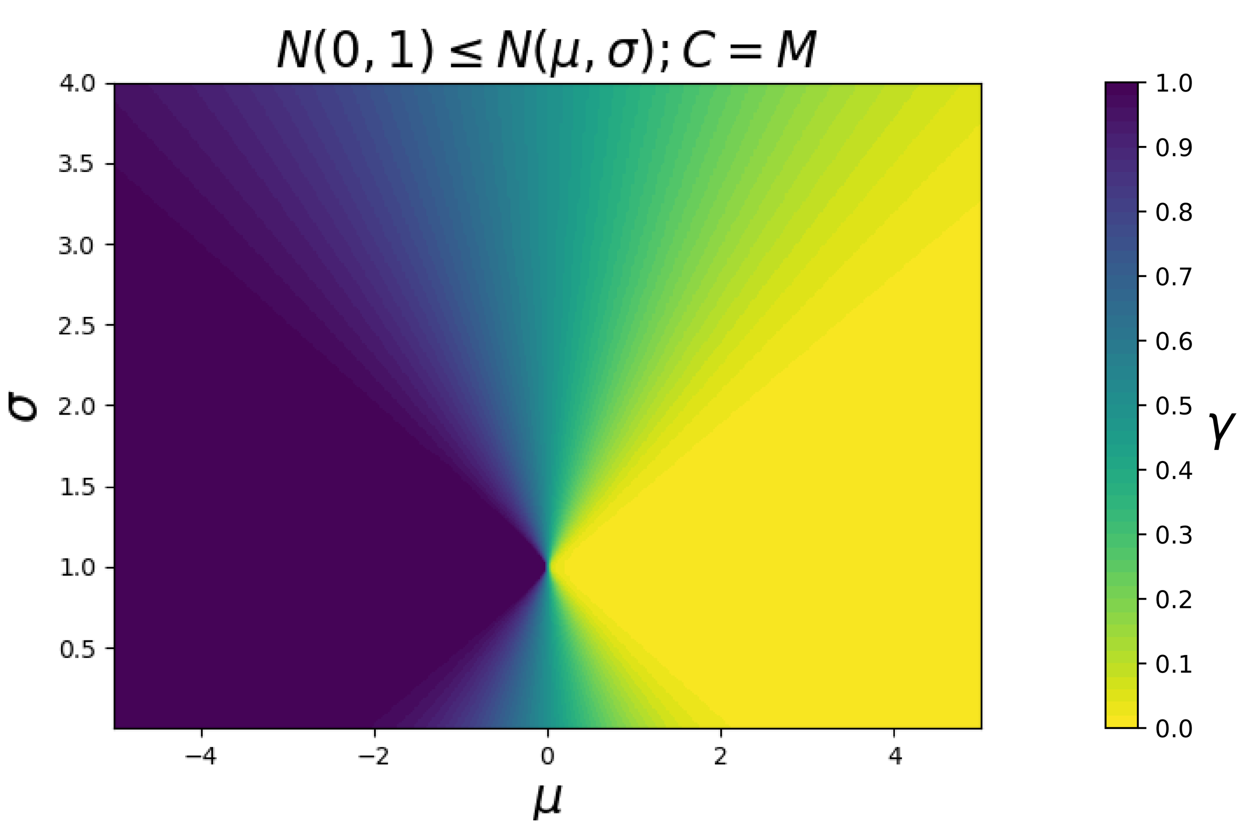


The so-called γ-index [1] is based on this interpretation of stochastic dominance by measuring how far a random variable is from stochastically dominating another. Formally, the γ-index is defined as follows:

$$\gamma(X,Y) = \ell\left(\{p \in]0,1[\mid F_X^{-1}(p) > F_Y^{-1}(p)\}\right),$$

where $\ell(\cdot)$ represents the Lebesgue measure. Note that $\gamma(X,Y) = 0$ means that $Y \succeq_{\text{FSD}} X$. If the quantile functions do not intersect, $\gamma(X,Y) = 1$ means that $X \succeq_{\text{FSD}} Y$.

The figure below represents the value of the γ-index $\gamma(X,Y)$, where $X \rightsquigarrow N(0,1)$ and $Y \rightsquigarrow N(\mu,\sigma)$. Yellow means values close to zero and blue means values close to one.



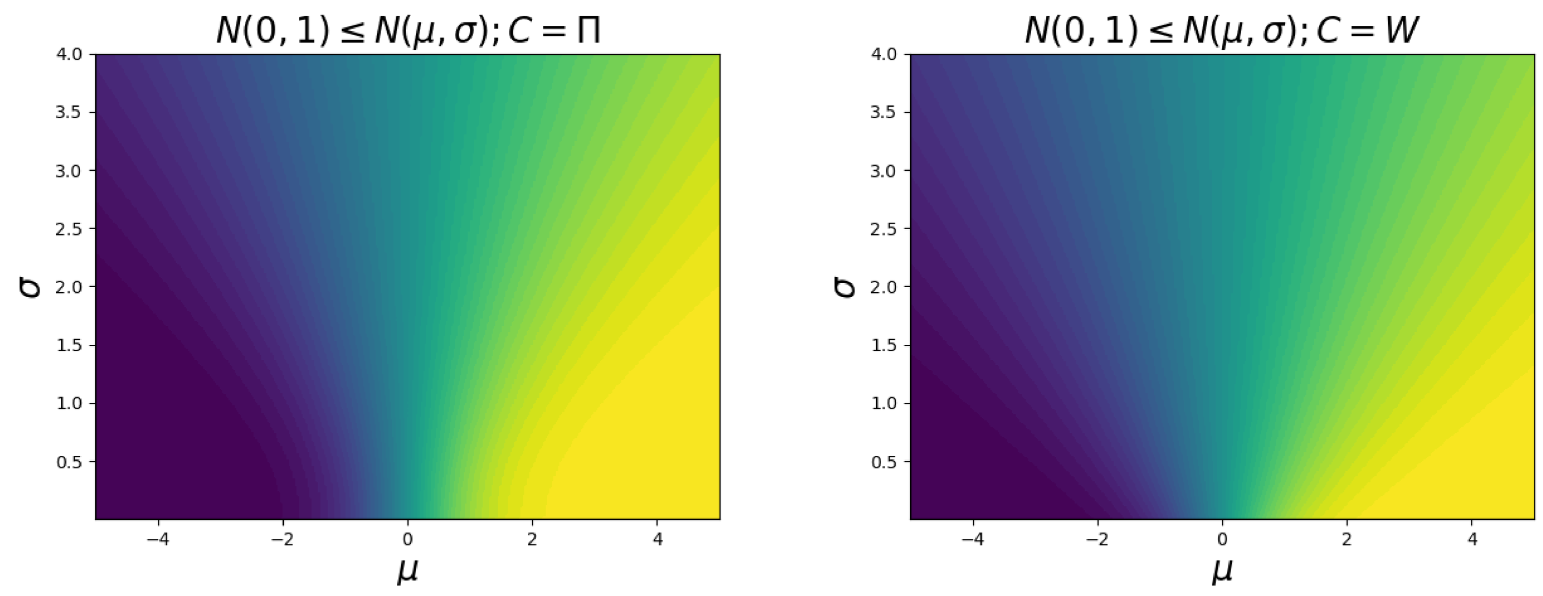
3. Taking dependence into account within the γ-index

Interestingly, under comonotonicity of X and Y (see [3]) and assuming $P(X = Y) = 0$, it holds that

$$\gamma(X,Y) = P(X \geq Y) = P(Y - X \leq 0) = F_{Y-X}(0).$$

This raises the question of whether we should take account of the dependence between the random variables when considering the γ-index. Indeed, the distribution of $Z = Y - X$ can be computed via the Lebesgue-Stieljes integral or convolution (see, e.g., [6]), by simply fixing the copula C modelling the dependence between X and Y.

The figures below represents the value of $P(X \geq Y)$, where $X \rightsquigarrow N(0,1)$ and $Y \rightsquigarrow N(\mu,\sigma)$, assuming that X and Y are independent (left, $C = \Pi$) and opposite (right, $C = W$). Interestingly, the γ-index corresponds to the perfect ($C = M$) case.

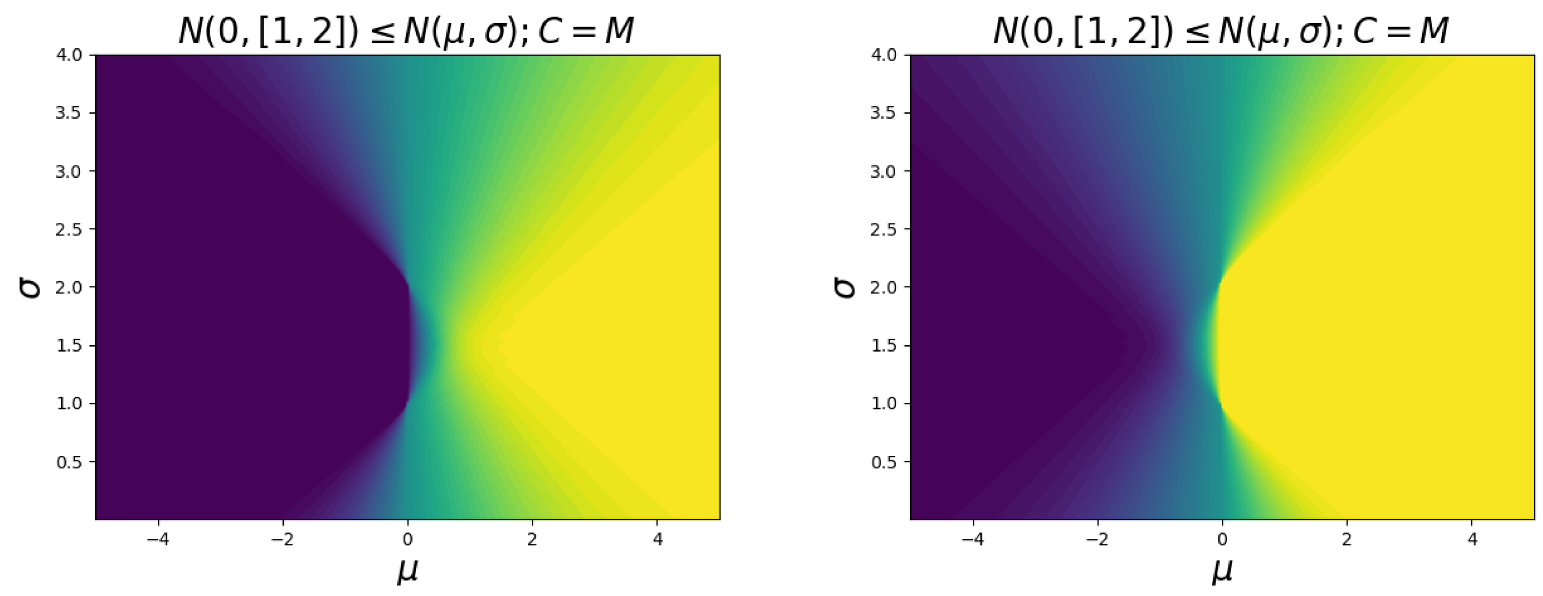


4. What happens under imprecision?

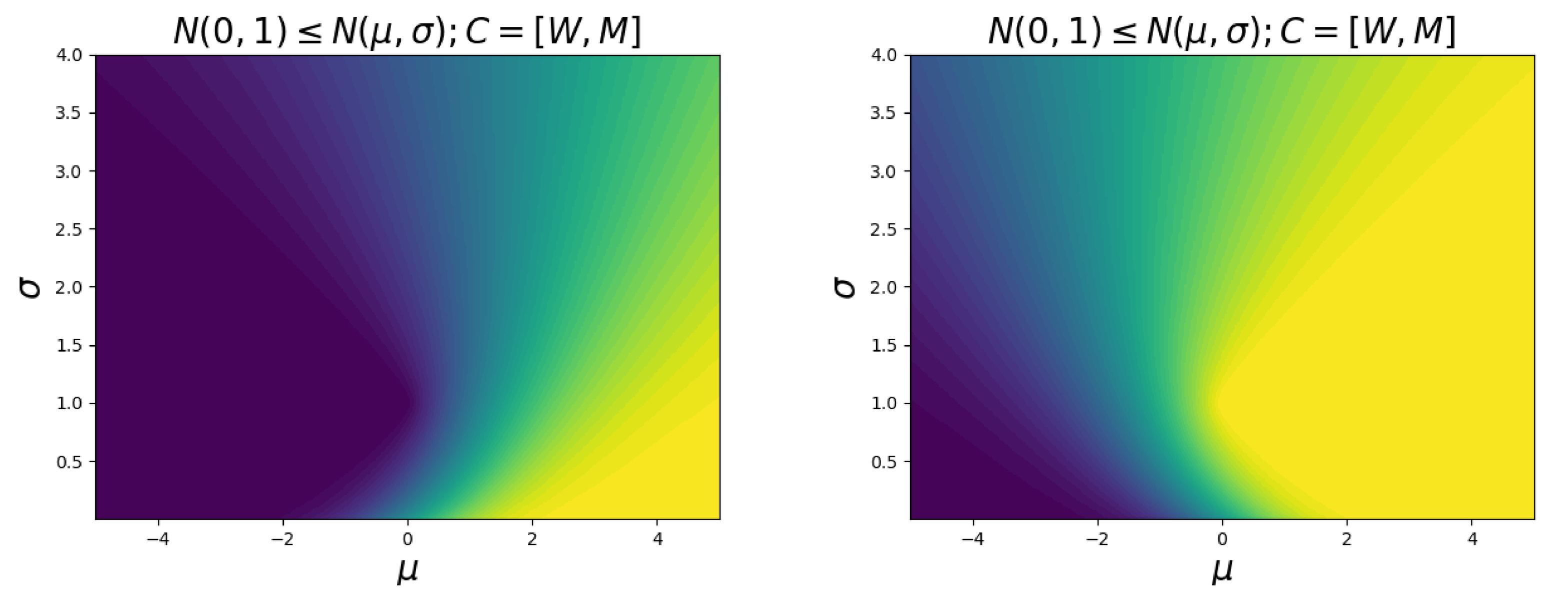
In some cases, there exists some imprecision on the distributions for which we would like to measure the deviation from stochastic dominance [4]. In such cases, we need to resort to interval arithmetic [6] and we would obtain an interval probability

$$[\underline{P}(X \geq Y), \overline{P}(X \geq Y)].$$

The figures below illustrate the lower bound (left) and the upper bound (right) for the case in which $X \rightsquigarrow N(0,[1,2])$ and $Y \rightsquigarrow N(\mu,\sigma)$, assuming that X and Y are comonotone.



The imprecision could also arise in the copula itself. For instance, the figures below illustrate the lower bound (left) and the upper bound (right) for the case in which $X \rightsquigarrow N(0,1)$ and $Y \rightsquigarrow N(\mu,\sigma)$, now considering that the copula is completely unknown, and using the Fréchet bounds ($C = [W, M]$). Note: the result is not a simple envelope of the two precise cases using M and W.



5. Conclusions

In this work, we have explored the use of the γ-index as a measure of deviation from ominance. Since there is annderlying assumption of comonotonicity between the random variables, the use of other types of dependences between the random variables has been modelled by means of copulas. It has been shown that the presented results are easily extended to the case in which there exists imprecision.

6. References

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