A Generalized Notion of Conjunction for Two Conditional Events

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Abstract

Traditionally the conjunction of conditional events has been defined as a *three-valued* object. However, in this way classical logical and probabilistic properties are not preserved. In recent literature, the conjunction of two conditional events $(A|H) \wedge (B|K)$ defined as a five-valued object with set of possible values $\{1, 0, x, y, z\}$, where $x = P(A|H), y = P(B|K)$, and $z = \mathbb{P}[(A|H) \wedge (B|K)]$ and satisfying classical probabilistic properties, has been deepened in the setting of coherence. In our paper we propose a generalization of this object, denoted by $(A|H) \wedge_{a,b} (B|K)$, where the values x and y are replaced by two arbitrary values $a, b \in [0, 1]$.

Preliminaries

Conditional events and random quantities

We denote by AH the conjunction of the events A and H. A conditional event $A|H$, with $H \neq \emptyset$, is looked at as a three-valued logical entity

$$A|H = \begin{cases} \text{True } (1), & \text{if } AH \text{ is true,} \\ \text{False } (0), & \text{if } \bar{A}H \text{ is true,} \\ \text{Void, } (x) & \text{if } \bar{H} \text{ is true,} \end{cases}$$

where $x = P(A|H)$. Given a random quantity X and an event $H \neq \emptyset$, we set

$$X|H = XH + \mathbb{P}(X|H)\bar{H}.$$

Definition 1 (Gilio & Sanfilippo 2014)

Given two conditional events $A|H$, $B|K$, with $P(A|H) = x$, $P(B|K) = y$, their conjunction is defined as $(A|H) \wedge (B|K) = (AHBK + x\bar{H}BK + yAH\bar{K})|(H \vee K)$, that is

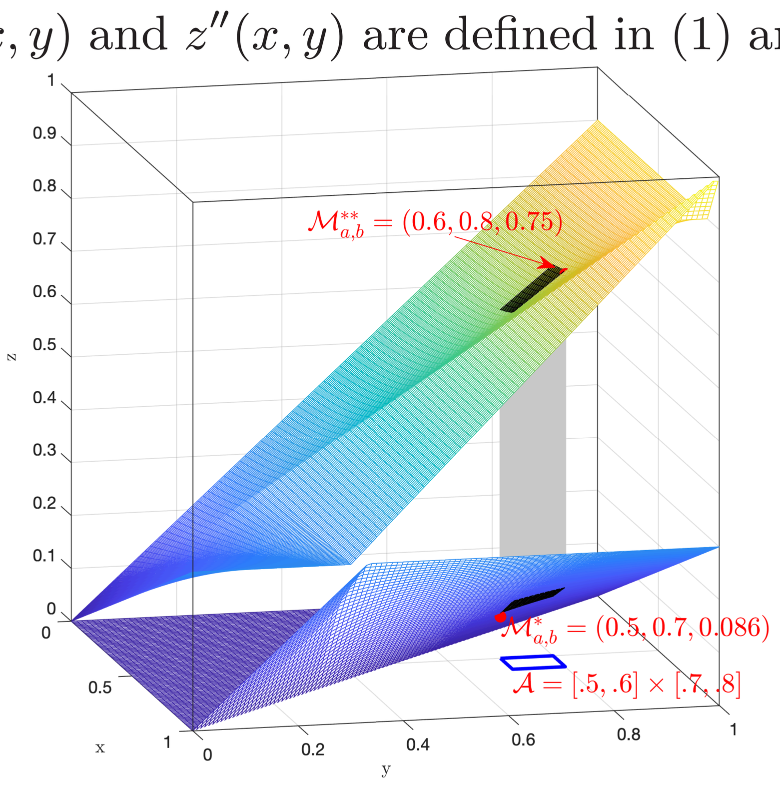
$$(A|H) \wedge (B|K) = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } \bar{A}H \vee \bar{B}K \text{ is true,} \\ x, & \text{if } \bar{H}BK \text{ is true,} \\ y, & \text{if } AH\bar{K} \text{ is true,} \\ z, & \text{if } \bar{H}\bar{K} \text{ is true,} \end{cases}$$

where $z = \mathbb{P}[(A|H) \wedge (B|K)]$. By coherence $z \in [\max\{x + y - 1, 0\}, \min\{x, y\}]$ (**F-H** bounds).

Imprecise Case

Theorem 2

Let $\mathcal{A} = ([x_1, x_2] \times [y_1, y_2])$ be an interval-valued assessment on $\{A|H, B|K\}$. Then, the interval of coherent extensions of $\mathcal{A}$ to $(A|H) \wedge_{a,b} (B|K)$ is the interval $[z^*, z^{**}] = [z'(x_1, y_1), z''(x_2, y_2)]$, where $z'(x, y)$ and $z''(x, y)$ are defined in (1) and (2), resp.



$\mathcal{A} = [.5, .6] \times [.7, .8]$, $[z^*, z^{**}] = [.086, .75]$

The Case $HK = \emptyset$

Theorem 3

Let an interval-valued probability assessment $\mathcal{A} = ([x_1, x_2] \times [y_1, y_2])$ on $\{A|H, B|K\}$, with $HK = \emptyset$, be given. Then, the interval of coherent extensions of $\mathcal{A}$ to $(A|H) \wedge_{a,b} (B|K)$ is $[z^*, z^{**}] = [\min\{ay_1, bx_1\}, \max\{ay_2, bx_2\}]$.

Full Paper!

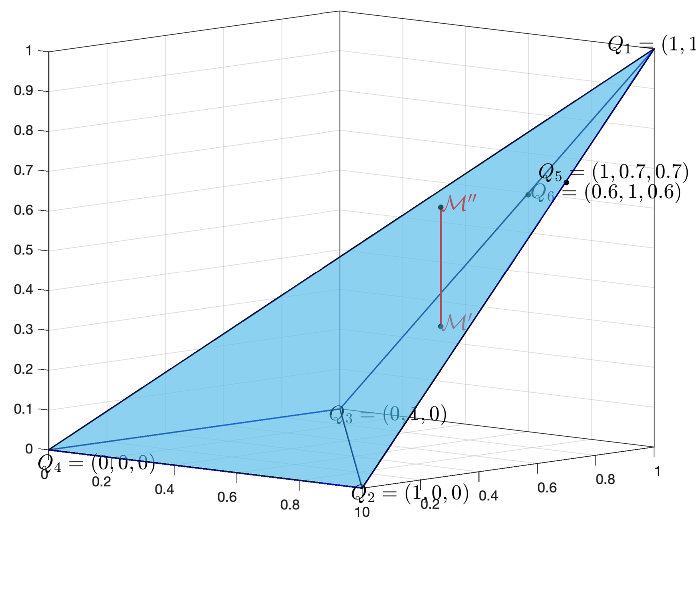
Main Result

Definition 2

Given four events A, B, H, K, with $H \neq \emptyset$ and $K \neq \emptyset$, and two values $a, b \in [0, 1]$, we define *the generalized conjunction w.r.t. a and b* of the conditional events $A|H$ and $B|K$ as the following conditional random quantity

$$(A|H) \wedge_{a,b} (B|K) = (AHBK + a\bar{H}BK + bAH\bar{K})|(H \vee K) = \begin{cases} 1 \text{ (win)}, & \text{if } A|H \text{ is true and } B|K \text{ is true} \\ 0 \text{ (lose)}, & \text{if } A|H \text{ is false or } B|K \text{ is false,} \\ a \text{ (partly win)}, & \text{if } A|H \text{ is void and } B|K \text{ is true,} \\ b \text{ (partly win)}, & \text{if } A|H \text{ is true and } B|K \text{ is void,} \\ z \text{ (called off)}, & \text{if } A|H \text{ is void and } B|K \text{ is void,} \end{cases}$$

where $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$.



$(a, b) = (x, y) = (.6, .7)$,
$\mathcal{M}' = (x, y, z') = (.6, .7, .3)$,
$\mathcal{M}''_{a,b} = (x, y, z'') = (.6, .7, .6)$

Theorem 1

Let A, B, H, K be any logically independent events. A prevision assessment $\mathcal{M} = (x, y, z)$ on the family of conditional random quantities $\mathcal{F} = \{A|H, (B|K), (A|H) \wedge_{a,b} (B|K)\}$ is coherent if and only if $(x, y) \in [0, 1]^2$ and $z \in [z', z'']$, where

$$z' = \begin{cases} (x + y - 1) \cdot \min\{\frac{a}{x}, \frac{b}{y}, 1\}, & \text{if } x + y - 1 > 0, \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

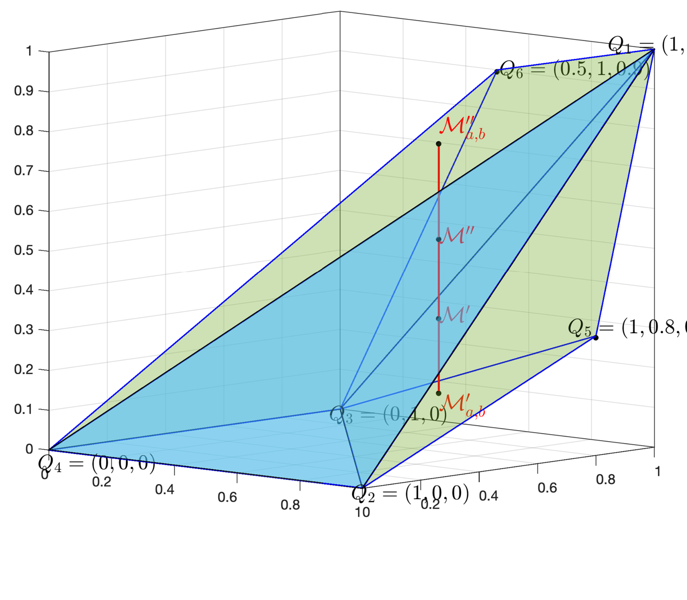
and

$$z'' = \max\{z''_1, z''_2, \min\{z''_3, z''_4\}\}, \quad (2)$$

where

$$z''_1 = \min\{x, y\}, \quad z''_2 = \begin{cases} \frac{x(b - ay) + y(a - bx)}{1 - xy}, & \text{if } (x, y) \neq (1, 1), \\ 1, \text{ if } (x, y) = (1, 1), \end{cases}$$

$$z''_3 = \begin{cases} \frac{x(1 - a) + y(a - x)}{1 - x}, & \text{if } x \neq 1, \\ 1, \text{ if } x = 1, \end{cases} \quad z''_4 = \begin{cases} \frac{x(b - y) + y(1 - b)}{1 - y}, & \text{if } y \neq 1, \\ 1, \text{ if } y = 1. \end{cases}$$



$(a, b) = (.9, .3)$,
$\mathcal{M}'_{a,b} = (x, y, z') = (.6, .7, .129)$,
$\mathcal{M}''_{a,b} = (x, y, z'') = (.6, .7, .675)$

Further aspects

Remark. When we assess $P(A|H) = x$ and $P(B|K) = y$, from definitions 1 and 2 it holds that

$$(A|H) \wedge_{x,y} (B|K) = (A|H) \wedge (B|K),$$

that is $(A|H) \wedge_{a,b} (B|K)$ reduces to $(A|H) \wedge (B|K)$ when $a = x$ and $b = y$. Moreover,
$\mathbb{P}[(A|H) \wedge_{x,y} (B|K)] = P(AHBK|(H \vee K)) + P(A|H)P(\bar{H}BK|(H \vee K)) + P(B|K)P(AH\bar{K}|(H \vee K))$.

Interpretation

Let us consider two individuals O and O'. Suppose that O' asserts $P'(A|H) = a$ and $P'(B|K) = b$. Then,

$$(A|H) \wedge_{a,b} (B|K) \overset{\text{Def. 2}}{=} (AHBK + a\bar{H}BK + bAH\bar{K})|(H \vee K) \overset{\text{Def. 1}}{=} (A|H) \wedge' (B|K),$$

where $(A|H) \wedge' (B|K)$ denotes the conjunction, as in Def.1, w.r.t. O'. Thus, $\mathbb{P}'[(A|H) \wedge_{a,b} (B|K)]$ satisfies the Fréchet-Hoeffding, that is:

$$\mathbb{P}'[(A|H) \wedge_{a,b} (B|K)] = \mathbb{P}'[(A|H) \wedge' (B|K)] \in [\max\{a + b - 1, 0\}, \min\{a, b\}].$$

Now, suppose that O asserts $P(A|H) = x$ and $P(B|K) = y$. Then, for the individual O, the lower and upper bounds z' and z'' on $(A|H) \wedge_{a,b} (B|K)$ computed by Theorem 1, represent the lower and upper bounds for the coherent extension $\mathbb{P}[(A|H) \wedge' (B|K)]$ of the assessment (x, y) on $\{A|H, B|K\}$. Therefore,

$$\mathbb{P}[(A|H) \wedge_{a,b} (B|K)] = \mathbb{P}[(A|H) \wedge' (B|K)] \neq \mathbb{P}[(A|H) \wedge (B|K)] = \mathbb{P}[(A|H) \wedge_{x,y} (B|K)].$$