ISIPTA 2023

A Generalized Notion of Conjunction for Two Conditional Events

LYDIA CASTRONOVO, GIUSEPPE SANFILIPPO
UNIVERSITY OF PALERMO, ITALY

Oviedo , July 13, 2023

matematica e informatica @ unipa

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Subjective Probability
Conditionals and iterated conditionals
Knowledge representation
and much more ...

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Preliminaries

Definition

Given two events A and $H \neq \emptyset$, the conditional event $A \mid H$ is a three-valued logical entity which is *True* when AH is true, *False* when $\bar{A}H$ is true and *Void* when $\bar{H}$ is true.

Preliminaries

$$A|H = \begin{cases} 1 & AH \\ 0 & \bar{A}H \\ x & \bar{H} \end{cases} \qquad x = P(A|H)$$

Definition

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How to define the conjunction of conditional events?

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How to define the conjunction of conditional events?

THREE-VALUED LOGICS

Kleene-de Finetti, Lukasiewicz, Bochvar, Sobocinski

Preliminaries

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How to define the conjunction of conditional events?

Definition (McGee 1989, Gilio and Sanfilippo, 2014)

Given two conditional events $A\,|\,H$, $B\,|\,K$ and a coherent probability assessment $P(A\,|\,H) = x$, $P(B\,|\,K) = y$, the conjunction $(A\,|\,H) \wedge (B\,|\,K)$ is defined as the following conditional random quantity

$$(A\,|\,H) \wedge (B\,|\,K) = (AHBK + x\bar{H}BK + yAH\bar{K})\,|\,(H \vee K)$$

Conjunction

Definition

Given two conditional events $A \mid H$, $B \mid K$ and a coherent probability assessment $P(A \mid H) = x$, $P(B \mid K) = y$, the conjunction $(A \mid H) \wedge (B \mid K)$ is defined as the following conditional random quantity $(A \mid H) \wedge (B \mid K) = (AHBK + x\bar{H}BK + yAH\bar{K}) \mid (H \vee K)$, where $z = \mathbb{P}((A \mid H) \wedge (B \mid K))$.

$$(A|H) \wedge (B|K) = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } \bar{A}H \vee \bar{B}K \text{ is true,} \\ x, & \text{if } \bar{H}BK \text{ is true,} \\ y, & \text{if } AH\bar{K} \text{ is true,} \\ z, & \text{if } \bar{H}\,\bar{K} \text{ is true,} \end{cases}$$

Conjunction

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↖ it's five-valued!

Conjunction

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Satisfies all the logic and probabilistic properties of conjunction!

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satisfies all the logic and probabilistic properties of conjunction!

it's five-valued!

The probabilistic properties are consistent with the results obtained in the field of conditional Boolean algebras (Flaminio, Godo, Hosni 2020; Flaminio, Gilio, Godo, Sanfilippo 2023).

What if we replace $x = P(A|H)$ and $y = P(B|K)$ with two arbitrary $a, b \in [0,1]$?

Definition

Given four events A, B, H, K, with $H \neq \emptyset$ and $K \neq \emptyset$, and two values $a, b \in [0,1]$, we define *the generalized conjunction* w.r.t. a and b of the conditional events $A|H$ and $B|K$ as the following conditional random quantity

$$(A|H) \wedge_{a,b} (B|K) = (AHBK + a\overline{H}BK + bAH\overline{K})|(H \vee K).$$

That is

$$(A|H) \wedge_{a,b} (B|K) = (AHBK + a\bar{H}BK + bAH\bar{K})|(H \vee K) =$$

$$= \begin{cases} 1 \text{ (win)}, & \text{if } A|H \text{ is true and } B|K \text{ is true} \\ 0 \text{ (lose)}, & \text{if } A|H \text{ is false or } B|K \text{ is false}, \\ a \text{ (partly win)}, & \text{if } A|H \text{ is void and } B|K \text{ is true}, \\ b \text{ (partly win)}, & \text{if } A|H \text{ is true and } B|K \text{ is void}, \\ z \text{ (called off)}, & \text{if } A|H \text{ is void and } B|K \text{ is void}, \end{cases}$$

where $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$.

↑ prevision

What if we replace $x = P(A|H)$ and $y = P(B|K)$ with two arbitrary $a, b \in [0,1]$?

Definition (Brand new!)

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↑ prevision

Do we still have all the classical properties of conjunction?

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where $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$.

↑ prevision

Do we still have all the classical properties of conjunction?

Fréchet-Hoeffding bounds for the conjunction

$$[\max\{x+y-1,0\}, \min\{x,y\}]$$

where $x = P(A|H)$ **and** $y = P(B|K)$.

Theorem

Let A, B, H, K be any logically independent events. A prevision assessment $\mathscr{M} = (x, y, z)$ on the family of conditional random quantities $\mathscr{F} = \{A|H, B|K, (A|H) \wedge_{a,b} (B|K)\}$ is coherent if and only if $(x, y) \in [0,1]^2$ and $z \in [z', z'']$, where

$$z' = \begin{cases} (x+y-1) \cdot \min\{\frac{a}{x}, \frac{b}{y}, 1\}, \text{ if } x+y-1>0 \\ 0, \text{ otherwise;} \end{cases}$$

$$z'' = \max\{z_1'', z_2'', \min\{z_3'', z_4''\}\},$$

where

$$z_1'' = \min\{x, y\},$$

$$z_2'' = \begin{cases} \frac{x(b-ay)+y(a-bx)}{1-xy}, \text{ if } (x,y) \neq (1,1), \\ 1, \text{ if } (x,y) = (1,1), \end{cases}$$

$$z_3'' = \begin{cases} \frac{x(1-a)+y(a-x)}{1-x}, \text{ if } x \neq 1, \\ 1, \text{ if } x = 1, \end{cases}$$

$$z_4'' = \begin{cases} \frac{x(b-y)+y(1-b)}{1-y}, \text{ if } y \neq 1, \\ 1, \text{ if } y = 1. \end{cases}$$

Theorem

Let A, B, H, K be any logically independent events. A prevision assessment $\mathcal{M} = (x, y, z)$ on the family of conditional random quantities $\mathcal{F} = \{A|H, B|K, (A|H) \wedge_{a,b} (B|K)\}$ is coherent if and only if $(x, y) \in [0,1]^2$ and $z \in [z', z'']$, where

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$$z'' = \max\{z_1'', z_2'', \min\{z_3'', z_4''\}\},$$

Case	z'
(A): $x+y-1 \le 0$	0
(B): $x+y-1>0$ Sub-cases:	
$(B.1)$: $a \ge x$ and $b \ge y$	$x+y-1$
$(B.2)$: $a < x$ and $\frac{a}{x} \le \frac{b}{y}$	$\frac{a}{x}(x+y-1)$
$(B.3)$: $b < y$ and $\frac{b}{y} \le \frac{a}{x}$	$\frac{b}{y}(x+y-1)$

where

$$z_1'' = \min\{x, y\},$$

$$z_2'' = \begin{cases} \frac{x(b-ay)+y(a-bx)}{1-xy}, \text{ if } (x,y) \neq (1,1), \\ 1, \text{ if } (x,y) = (1,1), \end{cases}$$

$$z_3'' = \begin{cases} \frac{x(1-a)+y(a-x)}{1-x}, \text{ if } x \neq 1, \\ 1, \text{ if } x = 1, \end{cases}$$

$$z_4'' = \begin{cases} \frac{x(b-y)+y(1-b)}{1-y}, \text{ if } y \neq 1, \\ 1, \text{ if } y = 1. \end{cases}$$

Case	z''
(C) $a(1-y)+b(1-x) > 1-xy$	$z_2'' = \frac{x(b-ay)+y(a-bx)}{1-xy}$
(D) $a(1-y)+b(1-x) \le 1-xy$ Sub-cases:	
$x \le y$ and $a \le x$	$z_1'' = x$
$x \le y$ and $a > x$	$z_3'' = \frac{x(1-a)+y(a-x)}{1-x}$
$y < x$ and $b \le y$	$z_1'' = y$
$y < x$ and $b > y$	$z_4'' = \frac{x(b-y)+y(1-b)}{1-y}$

Theorem

Let A, B, H, K be any logically independent events. A prevision assessment $\mathcal{M} = (x, y, z)$ on the family of conditional random quantities $\mathcal{F} = \{A|H, B|K, (A|H) \wedge_{a,b} (B|K)\}$ is coherent if and only if $(x, y) \in [0,1]^2$ and $z \in [z', z'']$, where

$$z' = \begin{cases} (x+y-1)\cdot \min\{\frac{a}{x}, \frac{b}{y}, 1\}, \text{ if } x+y-1>0 \\ 0, \text{ otherwise}; \end{cases}$$

$$z'' = \max\{z_1'', z_2'', \min\{z_3'', z_4''\}\},$$

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Case	z'
(A): $x+y-1 \leq 0$	0
(B): $x+y-1 > 0$	
Sub-cases:	
$(B.1)$: $a \geq x$ and $b \geq y$	$x+y-1$
$(B.2)$: $a < x$ and $\frac{a}{x} \leq \frac{b}{y}$	$\frac{a}{x}(x+y-1)$
$(B.3)$: $b < y$ and $\frac{b}{y} \leq \frac{a}{x}$	$\frac{b}{y}(x+y-1)$

Case	z''
(C)	
$a(1-y)+b(1-x) > 1-xy$	$z_2'' = \frac{x(b-ay)+y(a-bx)}{1-xy}$
(D)	
$a(1-y)+b(1-x) \leq 1-xy$	
Sub-cases:	
$x \leq y$ and $a \leq x$	$z_1'' = x$
$x \leq y$ and $a > x$	$z_3'' = \frac{x(1-a)+y(a-x)}{1-x}$
$y < x$ and $b \leq y$	$z_1'' = y$
$y < x$ and $b > y$	$z_4'' = \frac{x(b-y)+y(1-b)}{1-y}$

F-H bounds

Theorem (Convex-Hull of the points Q_i)

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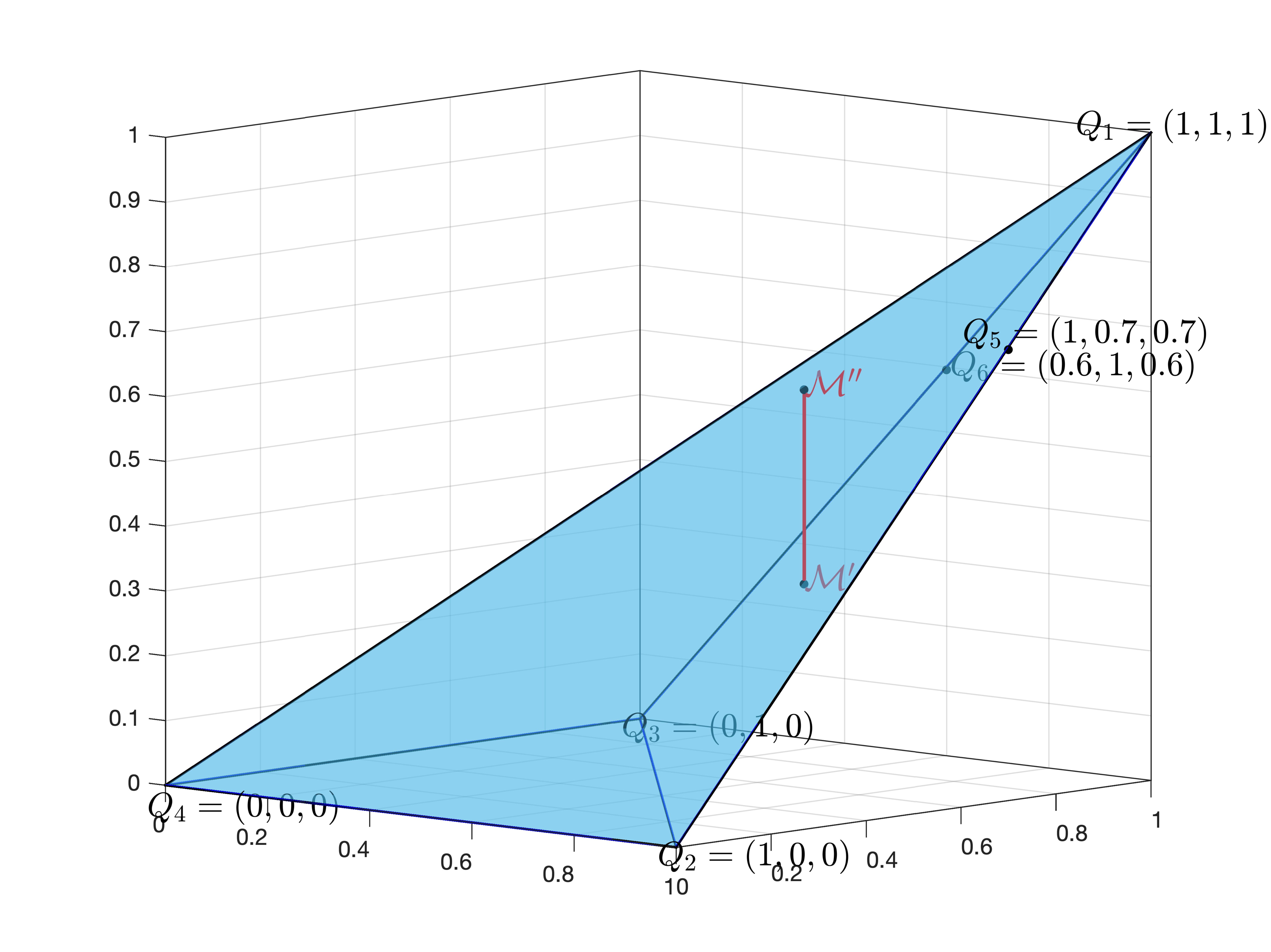
CASE a = x , b = y



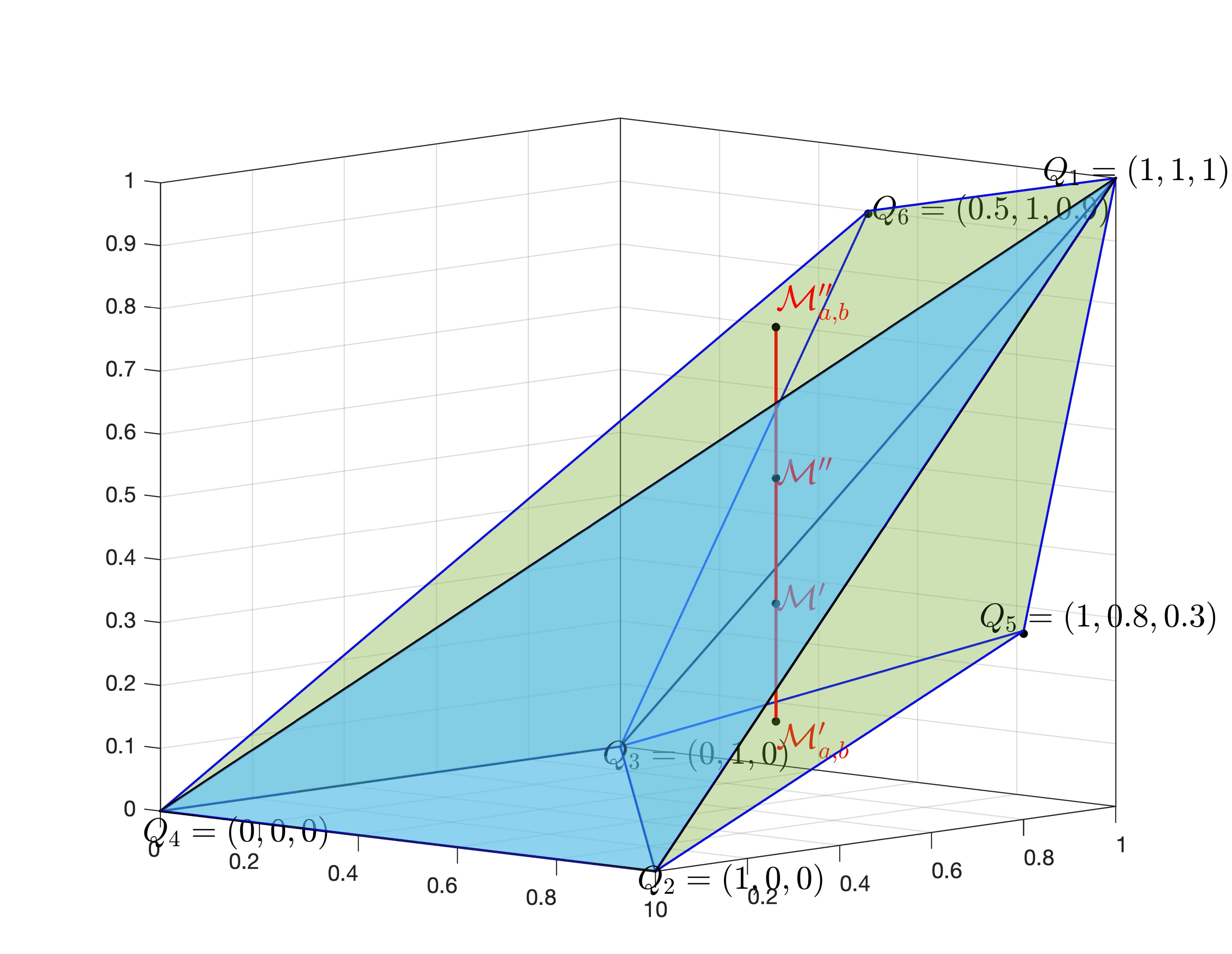
$(a, b) = (x, y) = (0.6, 0.7)$

$\mathcal{M}' = (x, y, z') = (0.6, 0.7, 0.3)$

$\mathcal{M}'' = (x, y, z'') = (0.6, 0.7, 0.6)$

Theorem (Convex-Hull of the points Q_i)

CASE $a \neq x$, $b \neq y$



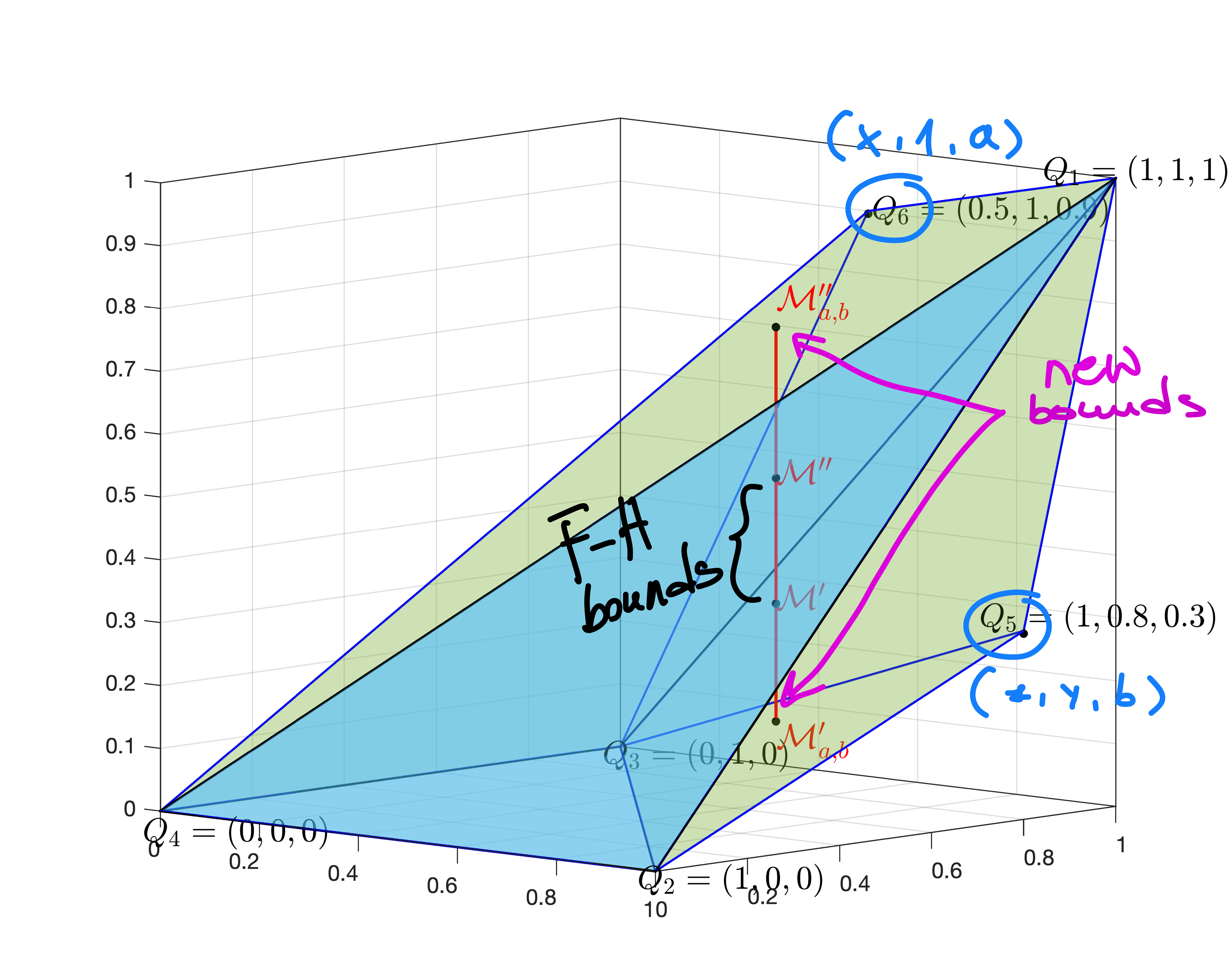
$(a, b) = (0.9, 0.3) \neq (x, y)$

$\mathcal{M}'_{a,b} = (x, y, z') = (0.6, 0.7, 0.129)$

$\mathcal{M}''_{a,b} = (x, y, z'') = (0.6, 0.7, 0.675)$

Theorem (Convex-Hull of the points Q_i)

CASE $a \neq x$, $b \neq y$



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What about specific values of a and b?

What about specific values of a and b?

Can we find already known conjunctions?

Quasi conjunction: a=b=1

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$$(A|H) \wedge_{1,1} (B|K) = (AHBK + \bar{H}BK + AH\bar{K})|(H \vee K) =$$

$$= \begin{cases} 1 \text{ (win)}, & \text{if } A|H \text{ is true and } B|K \text{ is true} \\ 0 \text{ (lose)}, & \text{if } A|H \text{ is false or } B|K \text{ is false}, \\ 1 \text{ (win)}, & \text{if } A|H \text{ is void and } B|K \text{ is true}, \\ 1 \text{ (win)}, & \text{if } A|H \text{ is true and } B|K \text{ is void}, \\ z \text{ (called off)}, & \text{if } A|H \text{ is void and } B|K \text{ is void}, \end{cases}$$

where $z = \mathbb{P}[(A \mid H) \wedge_{1,1} (B \mid K)]$.

(Gilio 2012)

Lower and Upper bounds

$$z'_S = \max\{x + y - 1, 0\}$$

$$z''_S = \begin{cases} \frac{x+y-2xy}{1-xy}, & \text{if } (x, y) \neq (1, 1) \\ 1, & \text{if } (x, y) = (1, 1). \end{cases}$$

Quasi conjunction: a=b=1

$$(A|H) \wedge_{1,1} (B|K) = (AHBK + \bar{H}BK + AH\bar{K})|(H \vee K) =$$

$$= \begin{cases} 1 \text{ (win)}, & \text{if } A|H \text{ is true and } B|K \text{ is true} \\ 0 \text{ (lose)}, & \text{if } A|H \text{ is false or } B|K \text{ is false}, \\ 1 \text{ (win)}, & \text{if } A|H \text{ is void and } B|K \text{ is true}, \\ 1 \text{ (win)}, & \text{if } A|H \text{ is true and } B|K \text{ is void}, \\ z \text{ (called off)}, & \text{if } A|H \text{ is void and } B|K \text{ is void}, \end{cases}$$

where $z = \mathbb{P}[(A|H) \wedge_{1,1} (B|K)]$.

(Gilio 2012)

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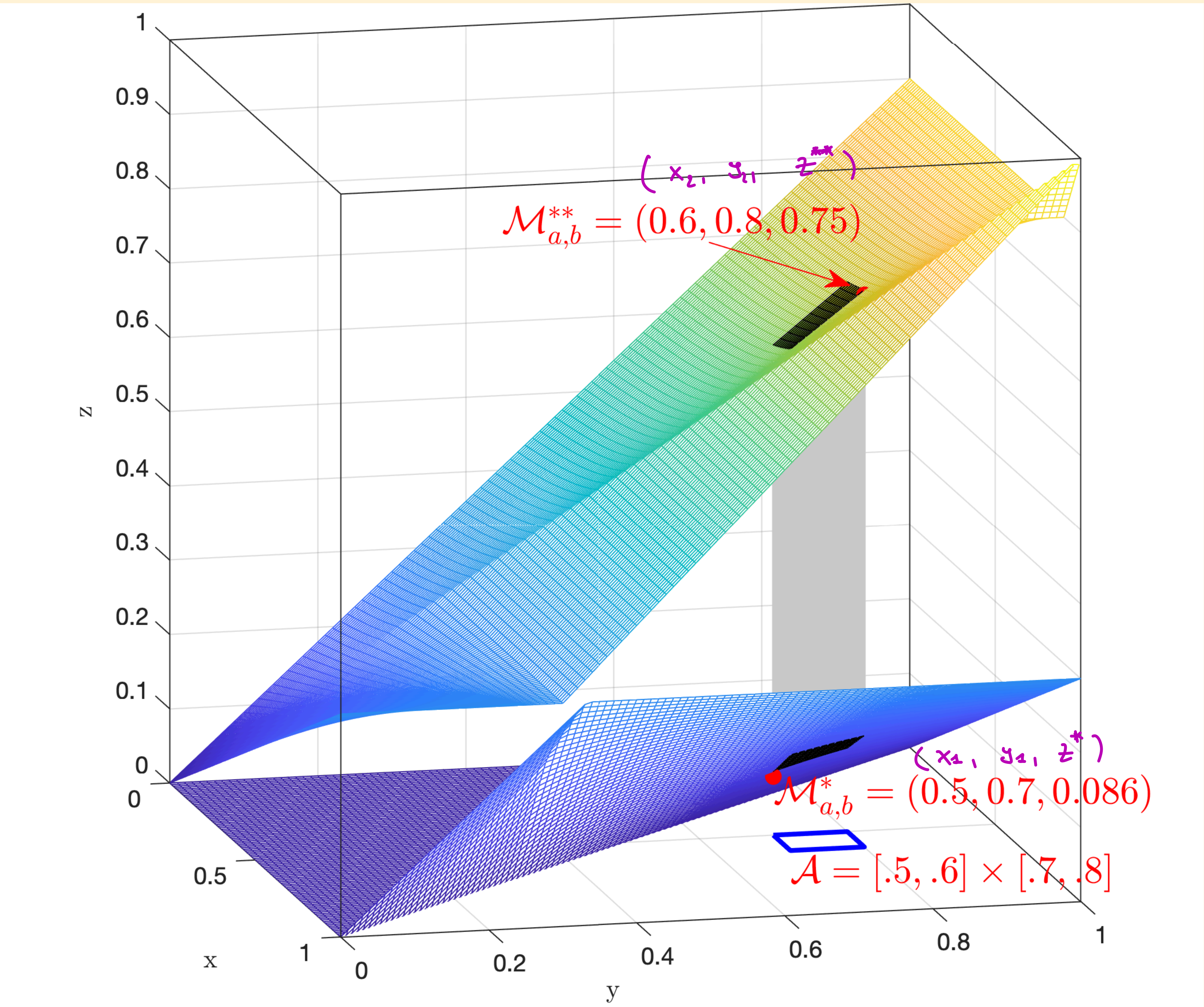
generalised conjunction

$$(A|H) \wedge_{(1,1)} (B|K) = (AHBK + \bar{H}BK + AH\bar{K})|(H \vee K) = (A|H) \wedge_S (B|K) =$$

$$= [(A \vee \bar{H}) \wedge (B \vee \bar{K})]|(H \vee K)$$

Sobocinsky conjunction (or quasi conjunction)

Imprecise Case



$(a, b) = (0.9, 0.3)$

Theorem:

Let A, B, H, K be any logically independent events and let $\mathscr{A} = ([x_1, x_2] \times [y_1, y_2])$ be an interval-valued assessment on $\{A|H, B|K\}$. Then, the interval of coherent extensions of $\mathscr{A}$ to $(A|H) \wedge_{a,b} (B|K)$ is the interval $[z^*, z^{**}] = [z'(x_1, y_1), z''(x_2, y_2)]$, where

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where

$$z''_1(x_2, y_2) = \min\{x_2, y_2\}, \qquad z''_2(x_2, y_2) = \begin{cases} \frac{x_2(b - a y_2) + y_2(a - b x_2)}{1 - x_2 y_2}, & \text{if } (x_2, y_2) \neq (1,1), \\ 1, & \text{if } (x_2, y_2) \neq (1,1), \end{cases}$$

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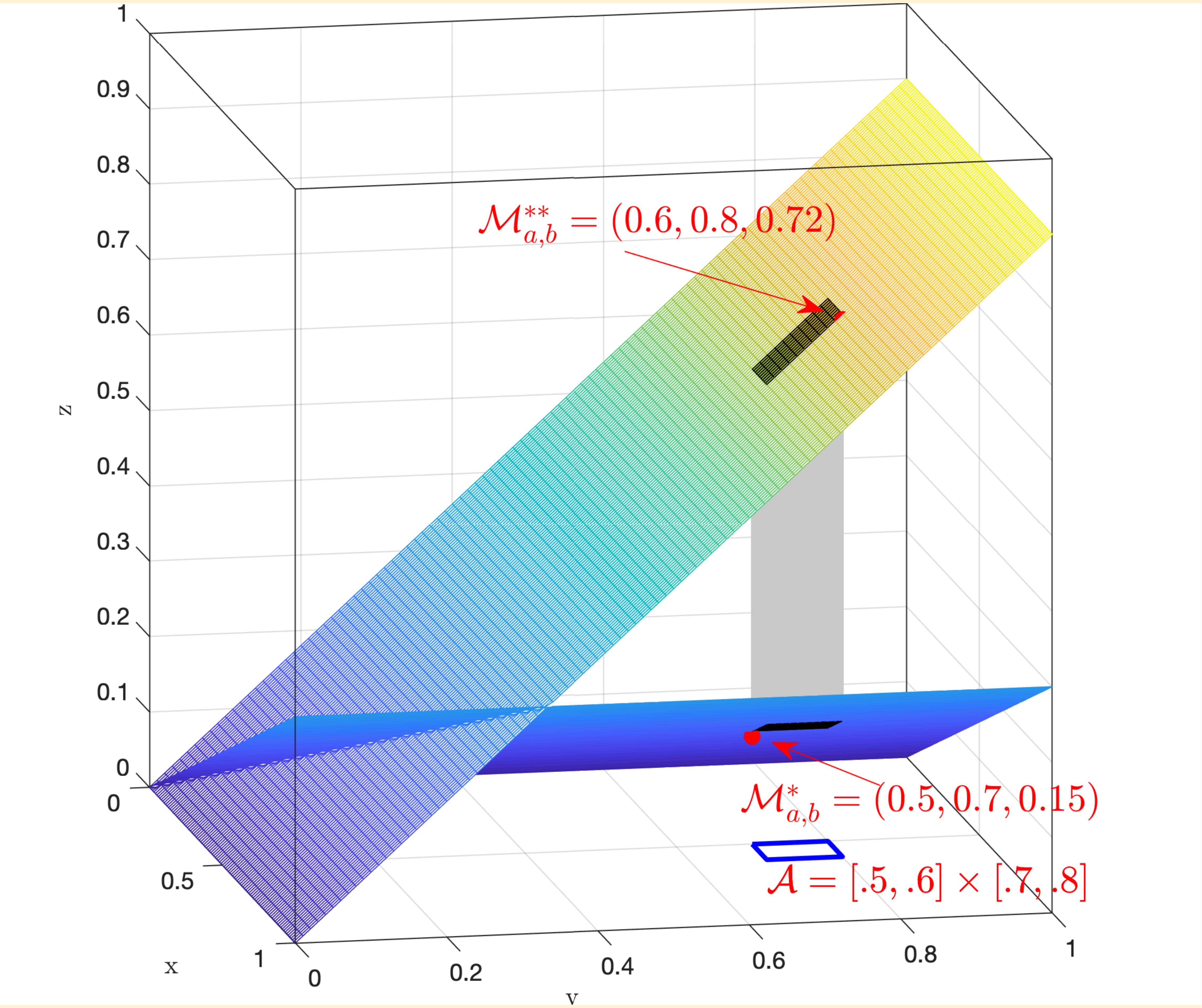
Imprecise Case: the case $HK = \emptyset$



$(a, b) = (0.9, 0.3)$

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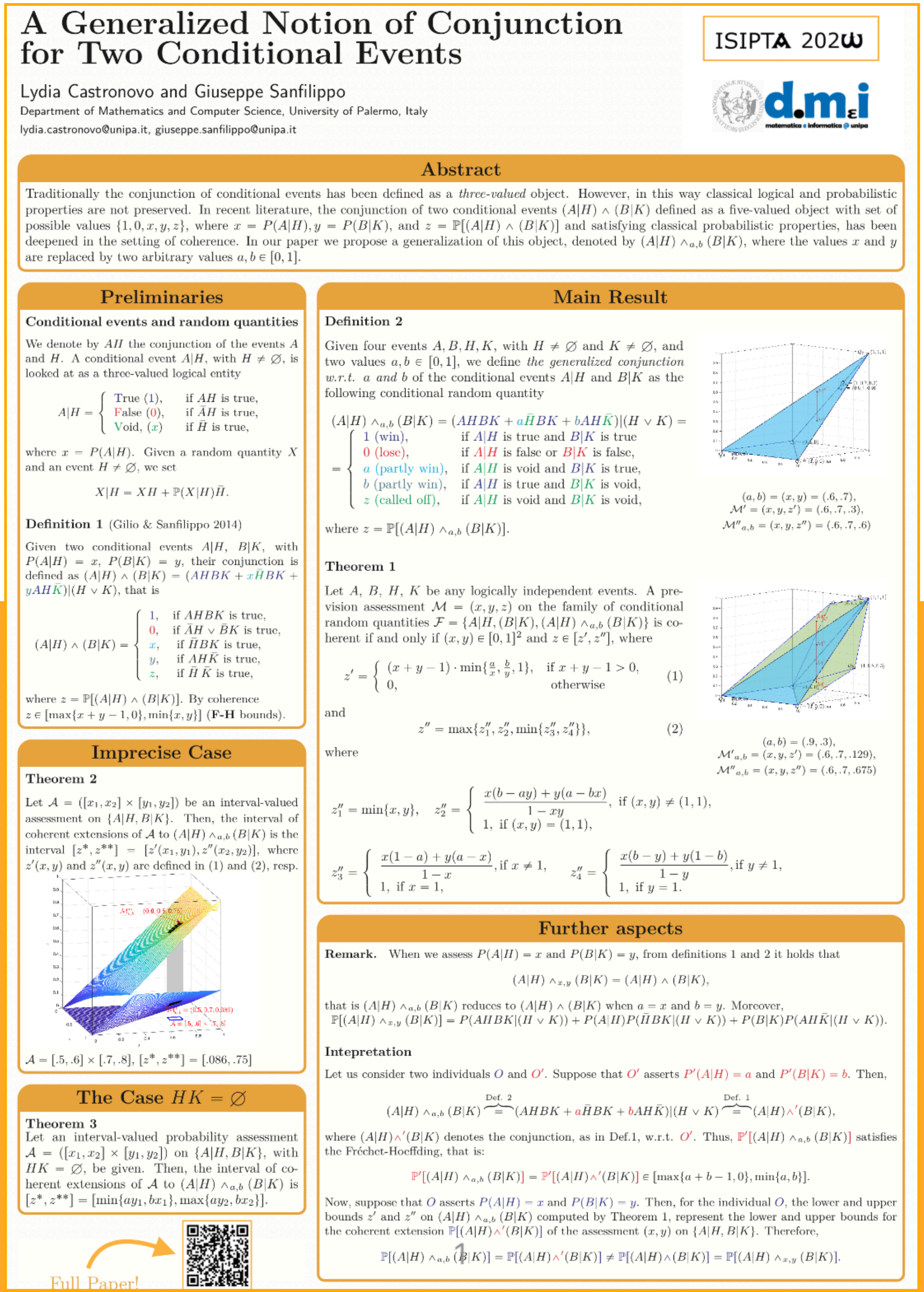

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ISIPTA 2021

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$\mathcal{A} = [.5, .6] \times [.7, .8]$, $[z^*, z^{**}] = [.086, .75]$

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Full Paper!

Main Result

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Given four events A, B, H, K, with $H \neq \emptyset$ and $K \neq \emptyset$, and two values $a, b \in [0,1]$, we define *the generalized conjunction w.r.t. a and b* of the conditional events $A|H$ and $B|K$ as the following conditional random quantity

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$(a, b) = (x, y) = (.6, .7)$,
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$$z' = \begin{cases} (x + y - 1) \cdot \min\{\frac{a}{x}, \frac{b}{y}, 1\}, & \text{if } x + y - 1 > 0, \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

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Further aspects

Remark. When we assess $P(A|H) = x$ and $P(B|K) = y$, from definitions 1 and 2 it holds that

$$(A|H) \wedge_{x,y} (B|K) = (A|H) \wedge (B|K),$$

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Let us consider two individuals O and O'. Suppose that O' asserts $P'(A|H) = a$ and $P'(B|K) = b$. Then,

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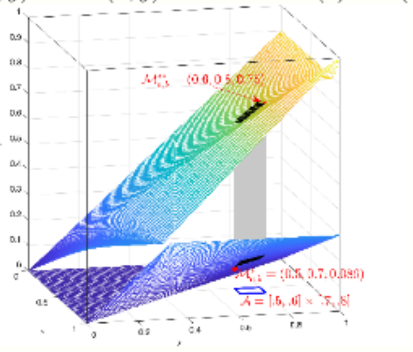

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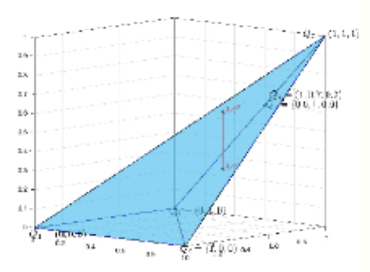

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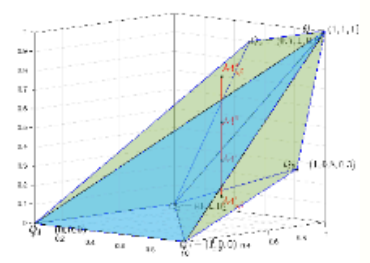

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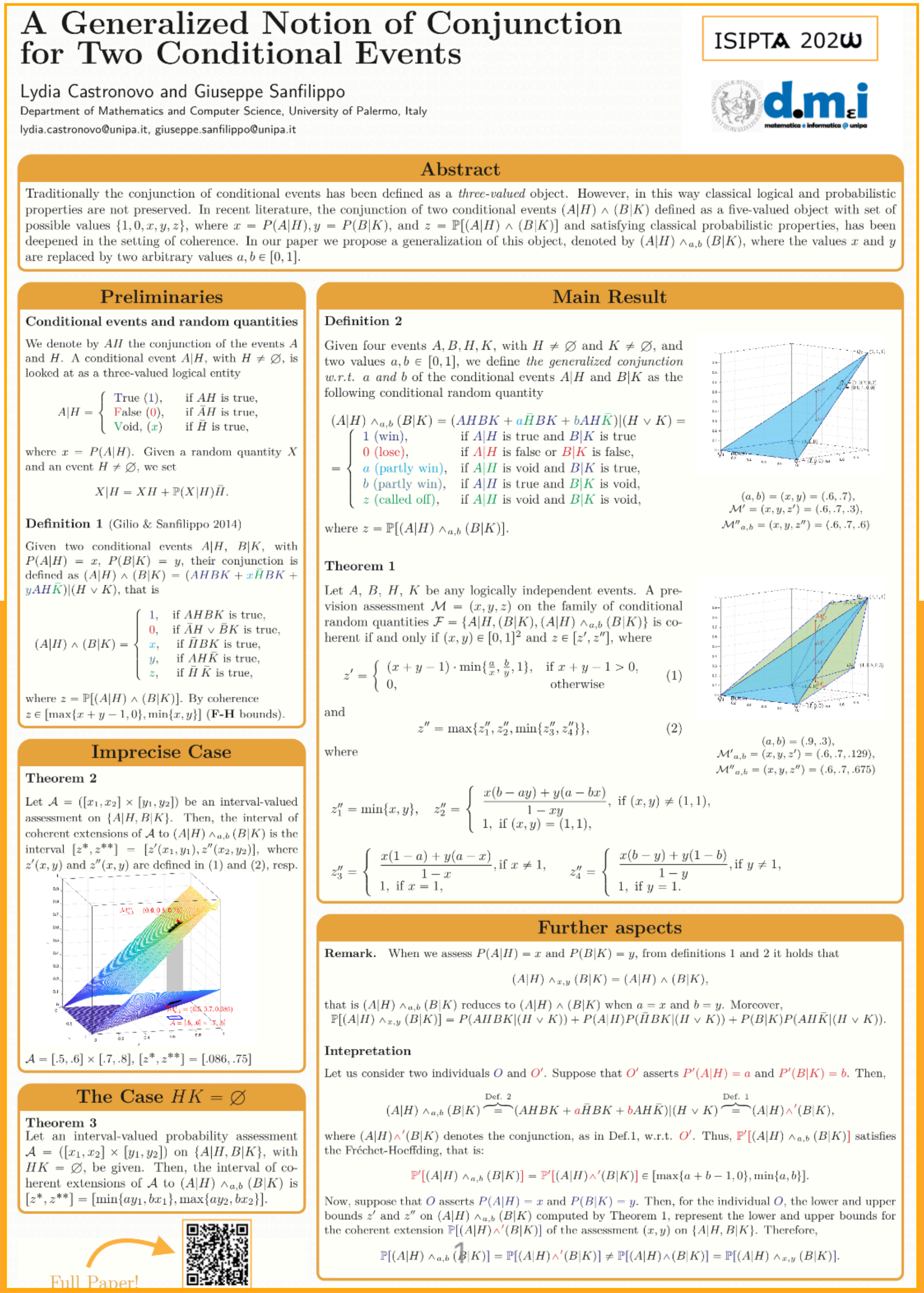

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Definition 2

Given four events A, B, H, K, with $H \neq \emptyset$ and $K \neq \emptyset$, and two values $a, b \in [0,1]$, we define *the generalized conjunction w.r.t. a and b* of the conditional events $A|H$ and $B|K$ as the following conditional random quantity

$$(A|H) \wedge_{a,b} (B|K) = (AHBK + a\bar{H}BK + bAH\bar{K})|(H \vee K) = \begin{cases} 1 \text{ (win)}, & \text{if } A|H \text{ is true and } B|K \text{ is true} \\ 0 \text{ (lose)}, & \text{if } A|H \text{ is false or } B|K \text{ is false,} \\ a \text{ (partly win)}, & \text{if } A|H \text{ is void and } B|K \text{ is true,} \\ b \text{ (partly win)}, & \text{if } A|H \text{ is true and } B|K \text{ is void,} \\ z \text{ (called off)}, & \text{if } A|H \text{ is void and } B|K \text{ is void,} \end{cases}$$

where $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$.

$(a,b) = (x,y) = (.6, .7)$,
$\mathcal{M}' = (x, y, z') = (.6, .7, .3)$,
$\mathcal{M}''_{a,b} = (x, y, z'') = (.6, .7, .6)$

Theorem 1

Let A, B, H, K be any logically independent events. A prevision assessment $\mathcal{M} = (x, y, z)$ on the family of conditional random quantities $\mathcal{F} = \{A|H, (B|K), (A|H) \wedge_{a,b} (B|K)\}$ is coherent if and only if $(x, y) \in [0,1]^2$ and $z \in [z', z'']$, where

$$z' = \begin{cases} (x+y-1) \cdot \min\{\frac{a}{x}, \frac{b}{y}, 1\}, & \text{if } x+y-1 > 0, \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

and

$$z'' = \max\{z_1'', z_2'', \min\{z_3'', z_4''\}\}, \quad (2)$$

where

$$z_1'' = \min\{x, y\}, \quad z_2'' = \begin{cases} \frac{x(b-ay) + y(a-bx)}{1-xy}, & \text{if } (x,y) \neq (1,1), \\ 1, & \text{if } (x,y) = (1,1), \end{cases}$$

$$z_3'' = \begin{cases} \frac{x(1-a) + y(a-x)}{1-x}, & \text{if } x \neq 1, \\ 1, & \text{if } x = 1, \end{cases} \quad z_4'' = \begin{cases} \frac{x(b-y) + y(1-b)}{1-y}, & \text{if } y \neq 1, \\ 1, & \text{if } y = 1. \end{cases}$$

$(a,b) = (.9, .3)$,
$\mathcal{M}'_{a,b} = (x, y, z') = (.6, .7, .129)$,
$\mathcal{M}''_{a,b} = (x, y, z'') = (.6, .7, .675)$

Further aspects

Remark. When we assess $P(A|H) = x$ and $P(B|K) = y$, from definitions 1 and 2 it holds that

$$(A|H) \wedge_{x,y} (B|K) = (A|H) \wedge (B|K),$$

that is $(A|H) \wedge_{a,b} (B|K)$ reduces to $(A|H) \wedge (B|K)$ when $a = x$ and $b = y$. Moreover, $\mathbb{P}[(A|H) \wedge_{x,y} (B|K)] = P(AHBK|(H \vee K)) + P(A|H)P(\bar{H}BK|(H \vee K)) + P(B|K)P(AH\bar{K}|(H \vee K))$.

Intepretation

Let us consider two individuals O and O'. Suppose that O' asserts $P'(A|H) = a$ and $P'(B|K) = b$. Then,

$$(A|H) \wedge_{a,b} (B|K) \overset{\text{Def. 2}}{=} (AHBK + a\bar{H}BK + bAH\bar{K})|(H \vee K) \overset{\text{Def. 1}}{=} (A|H) \wedge' (B|K),$$

where $(A|H) \wedge' (B|K)$ denotes the conjunction, as in Def.1, w.r.t. O'. Thus, $\mathbb{P}'[(A|H) \wedge_{a,b} (B|K)]$ satisfies the Fréchet-Hoeffding, that is:

$$\mathbb{P}'[(A|H) \wedge_{a,b} (B|K)] = \mathbb{P}'[(A|H) \wedge' (B|K)] \in [\max\{a+b-1, 0\}, \min\{a, b\}].$$

Now, suppose that O asserts $P(A|H) = x$ and $P(B|K) = y$. Then, for the individual O, the lower and upper bounds z' and z'' on $(A|H) \wedge_{a,b} (B|K)$ computed by Theorem 1, represent the lower and upper bounds for the coherent extension $\mathbb{P}[(A|H) \wedge' (B|K)]$ of the assessment (x, y) on $\{A|H, B|K\}$. Therefore,

$$\mathbb{P}[(A|H) \wedge_{a,b} (B|K)] = \mathbb{P}[(A|H) \wedge' (B|K)] \neq \mathbb{P}[(A|H) \wedge (B|K)] = \mathbb{P}[(A|H) \wedge_{x,y} (B|K)].$$

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