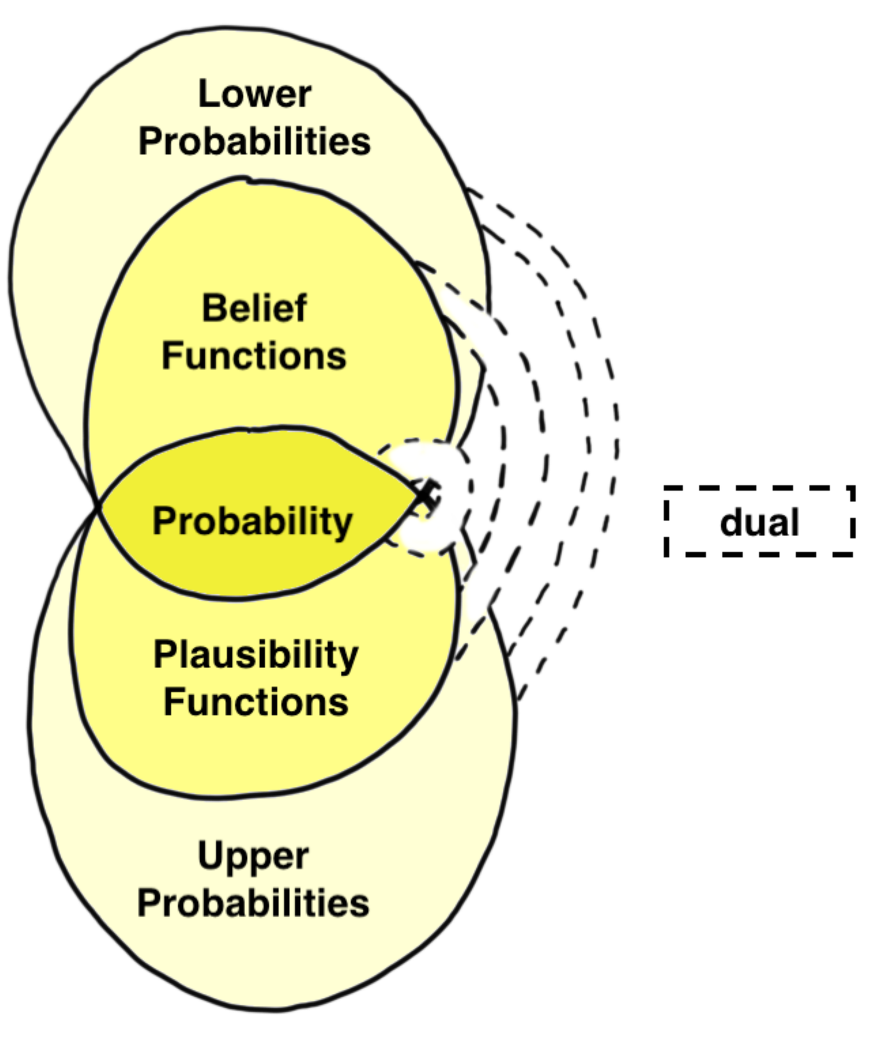
A modal logic for uncertainty: a completeness theorem

Esther Anna Corsi[1], Tommaso Flaminio[2], Lluís Godo[2], and Hykel Hosni[1]

[1]Department of Philosophy, University of Milan, Italy [2]Artificial Intelligence Research Institute (IIIA-CSIC), UAB, Spain

Uncertainty Measures



Łukasiewicz Logic

Axioms System:

$(L1)\ \varphi \to (\psi \to \varphi)$;

$(L2)\ (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$;

$(L3)\ (\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$;

$(L4)\ ((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$.

+ Modus Ponens

Standard MV-algebra:

$$[0,1]_{MV} = ([0,1], \oplus, \neg, 1)$$
$$a \oplus b = \min\{1, a+b\}$$
$$\neg a = 1 - a.$$

Standard Semantics:

$$e(\varphi \oplus \psi) = \min\{1, e(\varphi) + e(\psi)\};$$
$$e(\neg\varphi) = 1 - e(\varphi);$$
$$e(\varphi \wedge \psi) = \min\{e(\varphi), e(\psi)\};$$
$$e(\varphi \& \psi) = \max\{0, e(\varphi) + e(\psi) - 1\};$$
$$e(\varphi \vee \psi) = \max\{e(\varphi), e(\psi)\};$$
$$e(\varphi \to \psi) = \min\{1 - e(\varphi) + e(\psi), 1\}.$$

Modal Logic

S5 Kripke model:

Var : countable set of propositional variables

$\Box$: modal operator

$$\mathcal{K} = (W, R, \{e_w\}_{w \in W})$$

for every $w \in W$, e_w is a classical *evaluation*:

$$\|\varphi\|_{\mathcal{K},w} = e_w(\varphi)$$

$\|\Box\varphi\|_{\mathcal{K},w} = 1$ iff for each $w' \in W$, wRw' implies $e_{w'}(\varphi) = 1$

Axioms System for S5:

(CPL) Axioms of CPL;

$(K)\ \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$;

$(T)\ \Box\varphi \to \varphi$; + MP and $(N\Box)$

$(4)\ \Box\varphi \to \Box\Box\varphi$;

$(B)\ \varphi \to \Box\Diamond\varphi$.

S5Ł Kripke model: e_w evaluates Łukasiewicz (non-modal) formulas into the standard MV-algebra $[0,1]_{MV}$

$$\|\Box\varphi\|_{\mathcal{K},w} = \inf\{e_{w'}(\varphi) \mid wRw'\}.$$

Axioms System for S5Ł:

$(\Box 1)\ \Box\varphi \to \varphi$;

$(\Box 2)\ \Box(\psi \to \varphi) \to (\psi \to \Box\varphi)$;

$(\Box 3)\ \Box(\psi \vee \varphi) \to (\psi \vee \Box\varphi)$; + MP and $(N\Box)$

$(\Diamond 1)\ \Diamond(\varphi \& \varphi) \equiv \Diamond\varphi \& \Diamond\varphi$;

The language $\mathscr{L}_{\Box,P}$

$\mathscr{L}$: language of Łukasiewicz logic over finitely many (say n) propositional variables.

$\mathscr{L}_{\Box,P}$: the expansion of $\mathscr{L}$ by two additional unary modalities: $\Box$ and P.

CF: The set of *classical formulas*. Those are definable in $\mathscr{L}$ from variables, constants $\top$ and $\bot$, and the connectives $\wedge$, $\vee$, $\neg$.

Ex: φ, ψ, $\varphi \wedge \psi$

CMF: The set of *classical modal formulas* is defined by closing **CF** by the unary modality $\Box$ as usual in a modal language.

Ex: $\Box\varphi$, $\varphi \to \Box\psi$, $\Box\varphi \wedge \Diamond\psi$, $\Box(\Box\varphi \wedge \Diamond\psi)$

PMF: The set of *probabilistic modal formulas* is obtained by the following two steps:

1. *Atomic probabilistic formulas* are all those in the form $P(\varphi)$ for $\varphi \in$ **CMF**;

   Ex: $P(\varphi \to \Box\psi)$

2. *Compound probabilistic formulas* are defined by composing atomic ones with connectives of the Łukasiewicz language.

   Ex: $P(\varphi) \to P(\Box\psi)$

UMF: Finally, the set of *uncertainty modal formulas* is the smallest set of formulas that contains **PMF** and is closed under $\Box$ and connectives of Łukasiewicz logic.

Ex: $\Box P(\varphi)$, $\Box P(\varphi) \to P(\Diamond\psi)$, $\Box(P(\varphi) \to P(\Diamond\psi))$, $\Box(\Box P(\varphi) \to P(\Diamond\psi))$

LF: the set of *lower probability formulas* is the smallest subset of **UMF** that contains all basic formulas of the form $\Box P(\varphi)$ for every classical formula φ and that is closed under Łukasiewicz connectives.

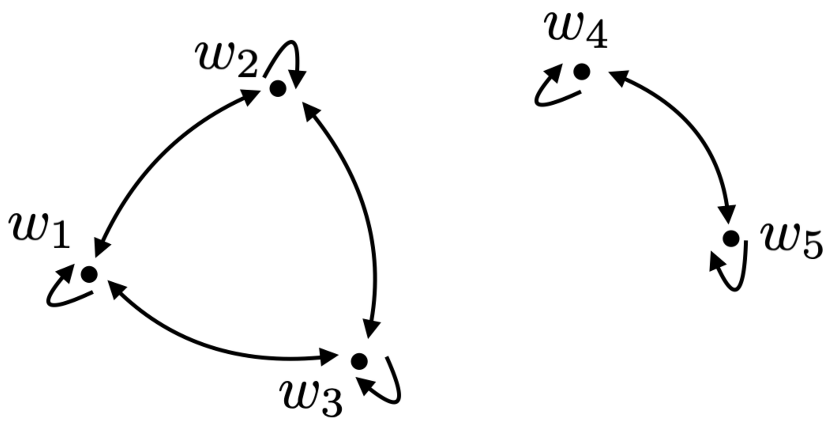
Ex: $\Box P\varphi \to \Box P\psi$ or $\neg\Box P\varphi$

Example

p, q, r: three propositional variables

$w_1 \models p, q, r;$
$w_2 \models p, \neg q, r;$
$w_3 \models \neg p, q, \neg r;$
$w_4 \models p, \neg q, \neg r$
$w_5 \models \neg p, \neg q, \neg r.$

	w_1	w_2	w_3	w_4	w_5
p_1	1/5	1/5	1/5	1/5	1/5
p_2	1/3	1/3	1/3	0	0
p_3	0	1/4	1/4	1/2	0
p_4	0	1/3	0	1/3	1/3
p_5	1/4	1/4	0	1/4	1/4



for all $i = 1, \ldots, 5$, $\quad \|P(\Box\varphi)\|_{\mathcal{U},w_i} = p_i(w_4) + p_i(w_5)$

$$\|\Box P(\varphi)\|_{\mathcal{U},w_1} = \min\{\|P(\varphi)\|_{\mathcal{U},w_1}, \|P(\varphi)\|_{\mathcal{U},w_2}, \|P(\varphi)\|_{\mathcal{U},w_3}\} = \min\{4/5, 2/3, 3/4\} = 2/3$$

	w_1	w_2	w_3	w_4	w_5
$\|P(\varphi)\|_{\mathcal{U},w_i}$	4/5	2/3	3/4	2/3	3/4
$\|\Box\varphi\|_{\mathcal{U},w_i}$	0	0	0	1	1
$\|P(\Box\varphi)\|_{\mathcal{U},w_i}$	2/5	0	1/2	2/3	1/2
$\|\Box P(\varphi)\|_{\mathcal{U},w_i}$	2/3	2/3	2/3	2/3	2/3

S5 Probability Models

An *S5 probability model* is a tuple

$$\mathcal{U} = (W, R, \{e_w\}_{w \in W}, \{p_w\}_{w \in W})$$

1. $(W, R, \{e_w\}_{w \in W})$ is a classical S5-Kripke model;
2. For all $w \in W$, p_w is a probability distribution on W and μ_w the probability function on 2^W induced by p_w.

(1) If $\varphi \in$ **CF**, then $\|\varphi\|_{\mathcal{U},w} = e_w(\varphi)$;

(2) If $\varphi \in$ **CMF**, then whenever $\varphi = \Box\psi$ we have

$$\|\varphi\|_{\mathcal{U},w} = \|\Box\psi\|_{\mathcal{U},w} = \inf\{\|\psi\|_{\mathcal{U},w'} \mid wRw'\}.$$

If φ is compound, then $\|\varphi\|_{\mathcal{U},w}$ is computed by truth-functionality using classical connectives.

(3) If $\varphi \in$ **PMF** and φ is atomic, i.e., $\varphi = P(\psi)$ with $\psi \in$ **CMF**, then

$$\|\varphi\|_{\mathcal{U},w} = \|P(\psi)\|_{\mathcal{U},w} = \sum\{p_w(w') \mid \|\psi\|_{\mathcal{U},w'} = 1\}.$$

If $\varphi \in$ **PMF** and is compound, then its truth-value is computed by truth-functionality using Łukasiewicz connectives.

(4) If $\varphi \in$ **UMF** and $\varphi = \Box\psi$, and thus with $\psi \in$ **PMF**, then

$$\|\varphi\|_{\mathcal{U},w} = \|\Box\psi\|_{\mathcal{U},w} = \inf\{\|\psi\|_{\mathcal{U},w'} \mid wRw'\}.$$

Ex: $\|\Box P(\psi)\|_{\mathcal{U},w} = \inf\{\|P(\psi)\|_{\mathcal{U},w'} \mid wRw'\} = \inf\{\mu_{w'}(\{w^* \in W \mid \|\psi\|_{\mathcal{U},w^*} = 1\}) \mid wRw'\}$

System S5(FP(Ł))

(CPL) The axioms and rules of classical propositional logic for formulas in **CF**.

(S5) The axioms of S5 applied to **CMF**.

(FP(Ł)) Axioms and rules of FP(Ł) for **PMF** formulas, i.e. the axioms of Ł the axioms for the modality P and Łukasiewicz implication:

$(P1)\ P(\varphi \to \psi) \to (P(\varphi) \to P(\psi))$;

$(P2)\ \neg P(\varphi) \equiv P(\neg\varphi)$;

$(P3)\ P(\varphi \vee \psi) \equiv [(P(\varphi) \to P(\varphi \wedge \psi)) \to P(\psi)]$;

(NP) necessitation: from φ infer $P(\varphi)$.

(S5(Ł)) Axioms and rules of S5(Ł) for **UMF** formulas.

Lower Probability Model

$$\mathcal{M} = (W, \{e_w\}_{w \in W}, \underline{P})$$

$\underline{P}$: lower probability on 2^W.

$$\|\Box P(\varphi)\|_{\underline{P}} = \underline{P}(\{w \mid e_w(\varphi) = 1\})$$

the evaluation extends to compound formulas in **LF** by truth-functionality using Łukasiewicz connectives.

Completeness Theorem

For every finite subset of formulas $T \cup \phi \subseteq (\mathbf{LF})$, the following conditions are equivalent:

(i) $T \vdash_{S5(FP(Ł))} \phi$

(ii) for all finite (universal) S5 probability model $\mathcal{U} = (W, R, \{e_w\}_{w \in W}, \{\mu_w\}_{w \in W})$ with $R = W \times W$, $\|\tau\|_{\mathcal{U}} = 1$ for each $\tau \in T$ implies $\|\phi\|_{\mathcal{U}} = 1$.

(iii) for all finite lower probability model $(W, \{e_w\}_{w \in W}, \underline{P})$, $\|\tau\|_{\underline{P}} = 1$ for each $\tau \in T$ implies $\|\phi\|_{\underline{P}} = 1$.