Desirable sets of things and their logic

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STONE representation for Boolean algebras

Choice functions and desirable gamble sets

Desirable sets of things

KRIPKE possible world semantics

Sets of desirable gambles

INSPIRATION FOR THIS WORK

Filters of probability measures

Desirable things

Consider a set of things T, some of which have an abstract property called desirability.

$S \subseteq T$ is a set of desirable things (SDT) to You if You state that *all things* in S desirable.

There is an inference mechanism associated with desirability via a *finitary* closure operator

$$\mathrm{Cl}_{\mathbf{D}} \colon \mathscr{P}(T) \to \mathscr{P}(T) \colon S \mapsto \mathrm{Cl}_{\mathbf{D}}(S).$$

D_1. if all things in S are desirable, then so are all things in $\mathrm{Cl}_{\mathbf{D}}(S)$.

There's a set of forbidden things T_-:

D_2. no thing in T_- is desirable.

The coherent SDTs are:

$$\overline{\mathbf{D}} := \{D \subseteq T \colon D = \mathrm{Cl}_{\mathbf{D}}(D) \text{ and } D \cap T_- = \emptyset\}.$$

Things in $T_+ := \mathrm{Cl}_{\mathbf{D}}(\emptyset)$ are always desirable.

D_3. $T_+ \cap T_- = \emptyset$, or equivalently, $\overline{\mathbf{D}} \neq \emptyset$.

CONJUNCTION

EMBEDDING

The structure $\langle \overline{\mathbf{D}}, \subseteq \rangle$ can be *embedded* in $\langle \overline{\mathbf{K}}_{\mathrm{fin}}, \subseteq \rangle$ by the endomorphism

$$D \mapsto K_D$$

with

$$K_D := \{S \subseteq T \colon D \cap S \neq \emptyset\}.$$

EMBEDDING

Desirable sets of things

$S \subseteq T$ is a desirable set of things to You if You state that *at least one thing* in S is desirable.

$K \subseteq \mathscr{P}(T)$ is Your set of desirable sets of things (SDS) if each $W \in K$ is a desirable set of things to You.

$\overline{\mathbf{K}}_{\mathrm{fin}}$ is the set (intersection structure) of all finitely coherent SDSes, and leads to a closure operator $\mathrm{Cl}_{\overline{\mathbf{K}}_{\mathrm{fin}}}$, defined by

$$\mathrm{Cl}_{\overline{\mathbf{K}}_{\mathrm{fin}}}(W) := \bigcap \{K \in \overline{\mathbf{K}}_{\mathrm{fin}} \colon W \subseteq K\}.$$

DISJUNCTION

LIFTING

An SDS $K \subseteq \mathscr{P}(T)$ is finitely coherent if:

K_1. $\emptyset \notin K$;

K_2. if $S_1 \in K$ and $S_1 \subseteq S_2$ then $S_2 \in K$, for all $S_1, S_2 \in \mathscr{P}(T)$;

K_3. if $S \in K$ then $S \setminus T_- \in K$, for all $S \in \mathscr{P}(T)$;

K_4. $\{t_+\} \in K$ for all $t_+ \in T_+$;

K_5. if $t_\sigma \in \mathrm{Cl}_{\mathbf{D}}(\sigma(W))$ for all $\sigma \in \Phi_W$, then $\{t_\sigma \colon \sigma \in \Phi_W\} \in K$, for all $\emptyset \neq W \Subset K$.

Here '$\Subset$' means 'is a finite subset of', and Φ_W is the set of all selection maps σ on W, so $\sigma(S) \in S$ for all $S \in W$.

LIFTING

Possible worlds models

You have a 'true' set of desirable things D_{T}, which assessments $W \Subset \mathscr{P}(T)$ provide information about. $\overline{\mathbf{D}}$ is a set of possible 'worlds'.

Each desirable set $S \in W$ leads to an event

$$D_S := \{D \in \overline{\mathbf{D}} \colon S \cap D \neq \emptyset\} \subseteq \overline{\mathbf{D}},$$

and the assessment $W \subseteq \mathscr{P}(T)$ to the event

$$\mathscr{E}(W) := \bigcap_{S \in W} D_S := \bigcap_{S \in W} \{D \in \overline{\mathbf{D}} \colon S \cap D \neq \emptyset\} \subseteq \overline{\mathbf{D}},$$

the set of all worlds that remain possible after Your assessment W.

The set of events $\mathbf{E}_{\mathrm{fin}} := \{\mathscr{E}(W) \colon W \Subset \mathscr{P}(T)\}$ is a bounded distributive lattice with top $\overline{\mathbf{D}}$ and bottom $\emptyset$.

Proper filters of events $\mathscr{F} \in \overline{\mathbb{F}}(\mathbf{E}_{\mathrm{fin}})$ correspond to consistent and deductively closed sets of propositional statements about D_{T}.

PROPOSITIONAL LOGIC

Syntax | Semantics

Complete SDSes

A finitely coherent SDS $K \in \overline{\mathbf{K}}_{\mathrm{fin}}$ is complete if

C. $(\forall S_1, S_2 \subseteq T)(S_1 \cup S_2 \in K \Rightarrow (S_1 \in K \text{ or } S_2 \in K))$.

$\overline{\mathbf{K}}_{\mathrm{fin,c}}$ is the set of all complete and finitely coherent SDSes.

The established order isomorphism allows us to translate the Prime Filter Representation Theorem into:

An SDS K is finitely coherent if and only if it is the *non-empty* intersection of all the complete and finitely coherent SDSes it is included in:

$$K = \underbrace{\bigcap \{K' \in \overline{\mathbf{K}}_{\mathrm{fin,c}} \colon K \subseteq K'\}}_{\neq \emptyset}.$$

REPRESENTATION

ORDER ISOMORPHISM

The structures $\langle \overline{\mathbf{K}}_{\mathrm{fin}}, \subseteq \rangle$ and $\langle \overline{\mathbb{F}}(\mathbf{E}_{\mathrm{fin}}), \subseteq \rangle$ are order isomorphic, via the order isomorphisms

$$\varphi^{\mathrm{fin}}_{\mathbf{D}}(K) := \{\mathscr{E}(W) \colon W \Subset K\},$$

and

$$\kappa^{\mathrm{fin}}_{\mathbf{D}}(\mathscr{F}) := \{S \subseteq T \colon \overline{\mathbf{D}}_S \in \mathscr{F}\}.$$

ORDER ISOMORPHISM

Prime filters

A proper filter $\mathscr{F} \in \overline{\mathbb{F}}(\mathbf{E}_{\mathrm{fin}})$ is prime if

PF. $(\forall E_1, E_2 \in \mathbf{E}_{\mathrm{fin}})(E_1 \cup E_2 \in \mathscr{F} \Rightarrow (E_1 \in \mathscr{F} \text{ or } E_2 \in \mathscr{F}))$.

$\overline{\mathbb{F}}_{\mathrm{p}}(\mathbf{E}_{\mathrm{fin}})$ is the set of all prime filters.

The well-known Prime Filter Representation Theorem states that:

A set of events $\mathscr{F}$ is a proper filter if and only if it is the *non-empty* intersection of all the prime filters it is included in:

$$\mathscr{F} = \underbrace{\bigcap \{\mathscr{G} \in \overline{\mathbb{F}}_{\mathrm{p}}(\mathbf{E}_{\mathrm{fin}}) \colon \mathscr{F} \subseteq \mathscr{G}\}}_{\neq \emptyset}.$$

REPRESENTATION

Finitary SDSes

We concentrate on the *finite sets of things* in

$$\mathscr{Q}(T) := \{S \in \mathscr{P}(T) \colon S \Subset T\}.$$

For any SDS $W \subseteq \mathscr{P}(T)$, we call

$$\mathrm{fin}(W) := W \cap \mathscr{Q}(T)$$

its finite part, and collect all its sets with finite desirable subsets in

$$\mathrm{fty}(W) := \{S \in \mathscr{P}(T) \colon (\exists \hat{S} \in W \cap \mathscr{Q}(T)) \hat{S} \Subset S\},$$

its finitary part.

An SDS $W \subseteq \mathscr{P}(T)$ is called finitary if all its desirable sets have finite desirable subsets, so

$$W \subseteq \mathrm{fty}(W).$$

A finitely coherent SDS K is finitary iff $K = \mathrm{fty}(K)$.

Conjunctive SDSes

A conjunctive SDS $W \subseteq \mathscr{P}(T)$ is a finitary SDS all of whose minimal elements are singletons:

$$(\forall S \in W)(\exists t \in S)\{t\} \in W.$$

A finitely coherent SDS K is conjunctive if and only if there is some coherent SDT D such that $K = K_D$ and then necessarily:

$$D = \{t \in T \colon \{t\} \in K\}.$$

The finitary part of any finitely coherent and complete SDS is finitely coherent and conjunctive; consequently, any finitary and finitely coherent SDS is complete if and only if it is conjunctive.

Note: the paper also discusses and studies stronger, infinitary versions of the lifting axioms K_1–K_5.

A *finitary* SDS K is finitely coherent if and only if it is the *non-empty* intersection of all the finitely coherent conjunctive SDSes it is included in:

$$K = \underbrace{\bigcap \{K_D \colon D \in \overline{\mathbf{D}} \text{ and } K \subseteq K_D\}}_{\neq \emptyset}.$$

REPRESENTATION