Indifference, symmetry and conditioning

Gert de Cooman

Foundations Lab (FLip)
Ghent University

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DESIRABILITY

Desirability: pioneers

PETER WILLIAMS

PETER WALLEY

TEDDY SEIDENFELD

Desirability: the basics

Options and preferences

The option space $\mathscr{U}$ is a real linear space, consisting of options u.

Desirability: the basics

EXAMPLES

- gambles $f\colon \mathscr{X} \to \mathbb{R}$ on some set $\mathscr{X}$
- indifference classes of gambles on some set $\mathscr{X}$
- Hermitian operators on a Hilbert space

Options and preferences

The option space $\mathscr{U}$ is a real linear space, consisting of options u.

A preference order $\rhd$ represents Your preferences between options:

$u \rhd v$ means that You strictly prefer option u over option v.

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Rationality criteria for preference

PR_1. the relation $\rhd$ is a strict partial preorder: irreflexive and transitive

PR_2. $u \rhd v \Rightarrow u + w \rhd v + w$ for all $u, v, w \in \mathscr{U}$

PR_3. $u \rhd v \Rightarrow \lambda u \rhd \lambda v$ for all $u, v \in \mathscr{U}$ and $\lambda > 0$

PR_4. if $u \succ v$ then also $u \rhd v$ for all $u, v \in \mathscr{U}$

Desirability: the basics

The background ordering $\succ$ is completely determined by its cone of positive options

$$\mathscr{U}_{\succ 0} := \{u \in \mathscr{U} : u \succ 0\}.$$

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Here, $\succ$ is some background preference order, reflecting those minimal preferences You must always have.

The preference order is typically partial, no totality requirement.

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The background ordering $\succ$ is completely determined by its cone of positive options

$$\mathscr{U}_{\succ 0} \coloneqq \{u \in \mathscr{U} : u \succ 0\}.$$

The preference order $\rhd$ is completely determined by the convex cone

$$D \coloneqq \{u \in \mathscr{U} : u \rhd 0\},$$

as

$$u \rhd v \Leftrightarrow u - v \rhd 0 \Leftrightarrow u - v \in D.$$

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Desirable options

A desirable option u is one You (strictly) prefer over the zero option.

We call D Your set of desirable options.

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Coherence criteria for desirability

D$_1$. $0 \notin D$

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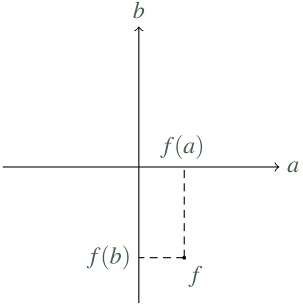
Desirability: the basics

$\mathscr{X} = \{a, b\}$



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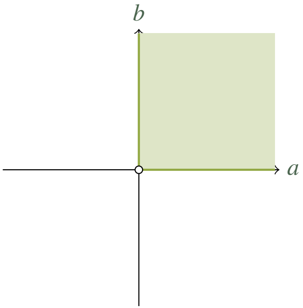

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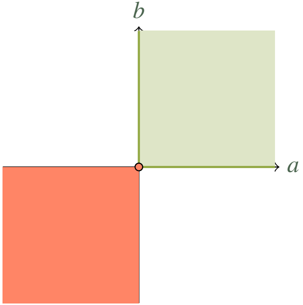

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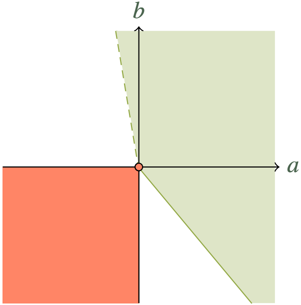

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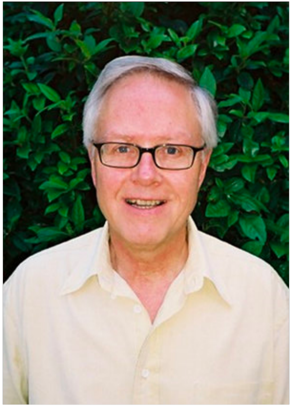
DERIVED ARCHIMEDEAN MODELS

Archimedean models: pioneers


BRUNO DE FINETTI


PETER WILLIAMS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Coherent and Archimedean choice in general Banach spaces

Gert de Cooman

Ghent University, Foundations Lab for Imprecise Probabilities, Technologiepark-Zwijnaarde 125, 9052 Zwijnaarde, Belgium

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ABSTRACT

I introduce and study a new notion of Archimedeanity for binary and non-binary choice between options that live in an abstract Banach space, through a very general class of choice models, called sets of desirable option sets. In order to be able to bring an important diversity of contexts into the fold, amongst which choice between horse lottery options, I pay special attention to the case where these linear spaces don't include all 'constant' options. I consider the frameworks of conservative inference associated with Archimedean (and coherent) choice models, and also pay quite a lot of attention to representation of general (non-binary) choice models in terms of the simpler, binary ones. The representation theorems proved here provide an axiomatic characterisation for, amongst many other choice methods, Levi's E-admissibility and Walley–Sen maximality.

Archimedean models: the basics

Structural assumptions

The option space $\mathscr{U}$, provided with a norm $\|\bullet\|_{\mathscr{U}}$, is a Banach space.

The norm $\|\bullet\|_{\mathscr{U}}$ induces a metric topology on $\mathscr{U}$, with interior operator Int and closure operator Cl.

A real functional $\Gamma\colon \mathscr{U} \to \mathbb{R}$ is bounded if its operator norm $\|\Gamma\|_{\mathscr{U}^\circ}$ is:

$$\|\Gamma\|_{\mathscr{U}^\circ} := \sup_{u \in \mathscr{U} \setminus \{0\}} \frac{|\Gamma(u)|}{\|u\|_{\mathscr{U}}} < +\infty.$$

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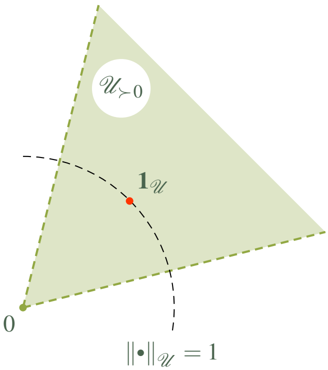

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$$\|\Gamma\|_{\mathscr{U}^\circ} := \sup_{u \in \mathscr{U} \setminus \{0\}} \frac{|\Gamma(u)|}{\|u\|_{\mathscr{U}}} < +\infty.$$

Take as unit element $\mathbf{1}_{\mathscr{U}}$ any (normed) element in the interior of $\mathscr{U}_{\succ 0}$:

$\mathbf{1}_{\mathscr{U}} \in \mathrm{Int}(\mathscr{U}_{\succ 0})$ and optionally $\|\mathbf{1}_{\mathscr{U}}\|_{\mathscr{U}} = 1.$

Archimedean models: buying and selling price functionals

Other ways to characterise Your preferences?

Buying price functional:

$$\underline{\Lambda}_D(u) := \sup\{\alpha \in \mathbb{R} \colon u - \alpha \mathbf{1}_{\mathscr{U}} \in D\} \text{ for all } u \in \mathscr{U}$$

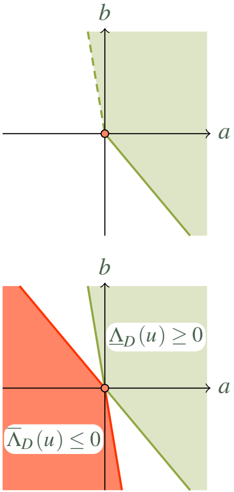
Selling price functional:

$$\overline{\Lambda}_D(u) := \inf\{\beta \in \mathbb{R} \colon \beta \mathbf{1}_{\mathscr{U}} - u \in D\} \text{ for all } u \in \mathscr{U}$$

Conjugacy:

$$\overline{\Lambda}_D(u) = -\underline{\Lambda}_D(-u) \text{ for all } u \in \mathscr{U}$$

Archimedean models: buying and selling price functionals



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Relation to Your preference model D

$$u \in \operatorname{Int}(D) \Leftrightarrow \underline{\Lambda}_D(u) > 0 \text{ and } u \in \operatorname{Cl}(D) \Leftrightarrow \underline{\Lambda}_D(u) \geq 0$$

The real functional $\underline{\Lambda}_D$ characterises D up to its topological boundary.

Archimedean models: coherent (lower and upper) previsions

Coherent lower prevision

A real functional $\underline{P}\colon \mathscr{U} \to \mathbb{R}$ is a coherent lower prevision if and only if there is some coherent set of desirable options D such that $\underline{P} = \underline{\Lambda}_D$.

Coherent upper prevision

A real functional $\overline{P}\colon \mathscr{U} \to \mathbb{R}$ is a coherent lower prevision if and only if there is some coherent set of desirable options D such that $\overline{P} = \overline{\Lambda}_D$.

Coherent prevision

A real functional $P\colon \mathscr{U} \to \mathbb{R}$ is a coherent prevision if and only if there is some coherent set of desirable options D such that $P = \underline{\Lambda}_D = \overline{\Lambda}_D$.

Archimedean models: coherent (lower and upper) previsions

Characterisation

A real functional $\underline{P}\colon \mathscr{U} \to \mathbb{R}$ is a coherent lower prevision if and only if

- LP$_1$. $\underline{P}(u+v) \geq \underline{P}(u) + \underline{P}(v)$ for all $u, v \in \mathscr{U}$
- LP$_2$. $\underline{P}(\lambda u) = \lambda \underline{P}(u)$ for all $u \in \mathscr{U}$ and all real $\lambda > 0$
- LP$_3$. $\|\underline{P}\|_{\mathscr{U}^\circ} < +\infty$
- LP$_4$. $\underline{P}(u + \alpha \mathbf{1}_{\mathscr{U}}) = \underline{P}(u) + \alpha$ for all $u \in \mathscr{U}$ and all real α
- LP$_5$. if $u \succ v$ then $\underline{P}(u) \geq \underline{P}(v)$ for all $u, v \in \mathscr{U}$

A real functional $P\colon \mathscr{U} \to \mathbb{R}$ is a coherent prevision if and only if

- P$_1$. $P(\lambda u + \mu v) = \lambda P(u) + \mu P(v)$ for all $u, v \in \mathscr{U}$ and all real λ, μ
- P$_2$. $\|P\|_{\mathscr{U}^\circ} < +\infty$
- P$_3$. $P(\mathbf{1}_{\mathscr{U}}) = 1$
- P$_4$. if $u \succ 0$ then $P(u) \geq 0$ for all $u \in \mathscr{U}$

INDIFFERENCE

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Accept & reject statement-based uncertainty models

Erik Quaeghebeur[a,b,*,1], Gert de Cooman[a], Filip Hermans[a]

[a] SYSTeMS Research Group, Ghent University, Technologiepark 914, 9052 Zwijnaarde, Belgium
[b] Centrum Wiskunde & Informatica, Postbus 94079, 1090 GB Amsterdam, The Netherlands

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ABSTRACT

We develop a framework for modelling and reasoning with uncertainty based on accept and reject statements about gambles. It generalises the frameworks found in the literature based on statements of acceptability, desirability, or favourability and clarifies their relative position. Next to the statement-based formulation, we also provide a translation in terms of preference relations, discuss—as a bridge to existing frameworks—a number of simplified variants, and show the relationship with prevision-based uncertainty models. We furthermore provide an application to modelling symmetry judgements.

Indifference: the basics

$u \equiv v$ expresses that You are indifferent between options u and v.

Rationality criteria for the indifference relation $\equiv$

I_1. the relation $\equiv$ is an equivalence relation: reflexive, symmetric and transitive;

I_2. $u \equiv v \Rightarrow u + w \equiv v + w$ for all $u, v, w \in \mathscr{U}$;

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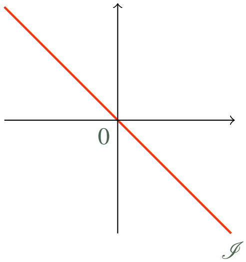

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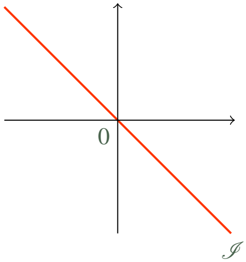
The indifference relation $\equiv$ is completely determined by the linear (sub)space

$$\mathscr{I} := \{u \in \mathscr{U} : u \equiv 0\},$$

as

$$u \equiv v \Leftrightarrow u - v \equiv 0 \Leftrightarrow u - v \in \mathscr{I}.$$

Indifference: the basics



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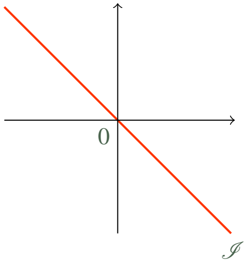
An indifferent option u is one You deem equivalent to the zero option.

We call $\mathscr{I}$ Your set of indifferent options.

Indifference: the basics



Desirability expresses a strict preference to the zero option.

Indifference expresses equivalence to the zero option.

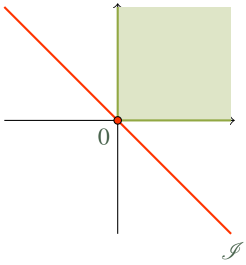
Desirability and indifference together

We call a set of desirable options D $\mathscr{I}$-compatible if

$$D + \mathscr{I} \subseteq D, \text{ or equivalently, } D + \mathscr{I} = D.$$

Adding an indifferent option to any option doesn't alter the latter's desirability.

Indifference: the basics



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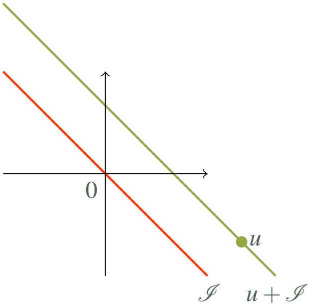
Compatibility condition

There are such $\mathscr{I}$-compatible and coherent sets of desirable options if and only if

$$\mathscr{I} \cap \mathscr{U}_{\succ 0} = \emptyset, \text{ or equivalently, } \mathscr{I} \cap \mathscr{U}_{\prec 0} = \emptyset.$$

Indifference: quotient spaces



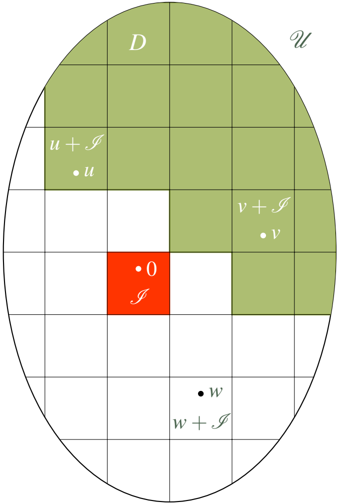
Equivalence classes under indifference

Partition the option space $\mathscr{U}$ into a collection of affine subspaces parallel to $\mathscr{I}$:

$$[u]_{\mathscr{I}} := u + \mathscr{I} = \{v \in \mathscr{U} : v \equiv u\}$$

is the set of all options that are indifferent to the option u.

Indifference: quotient spaces



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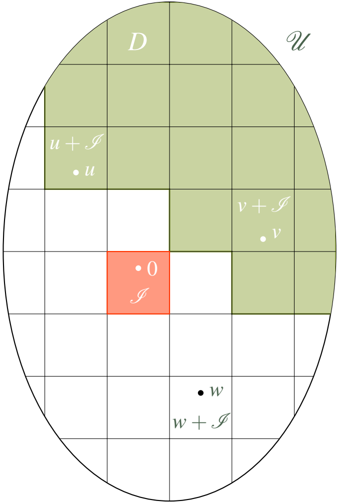
Crucial, if simple, observation

If $D + \mathscr{I} \subseteq D$ then

$$u \in D \Leftrightarrow [u]_{\mathscr{I}} \subseteq D \text{ for all } u \in \mathscr{U}.$$

Under indifference, desirability is a class property!

Indifference: the essence of representation



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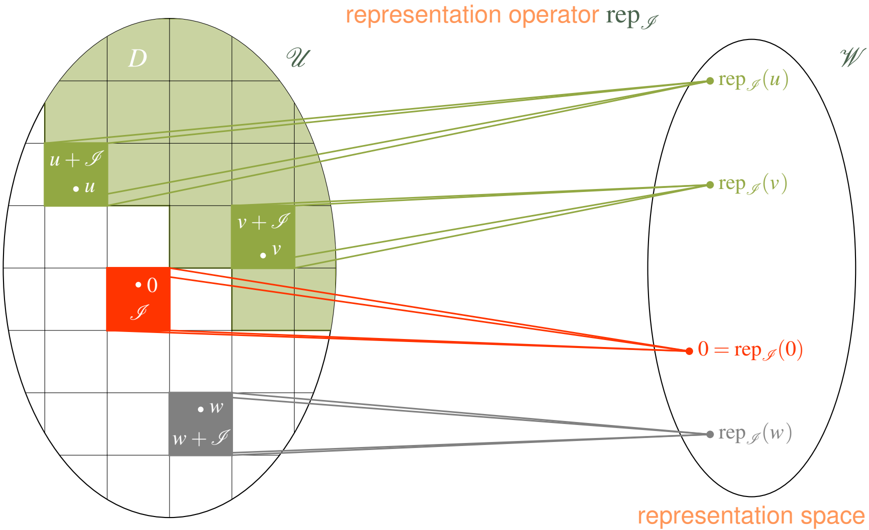

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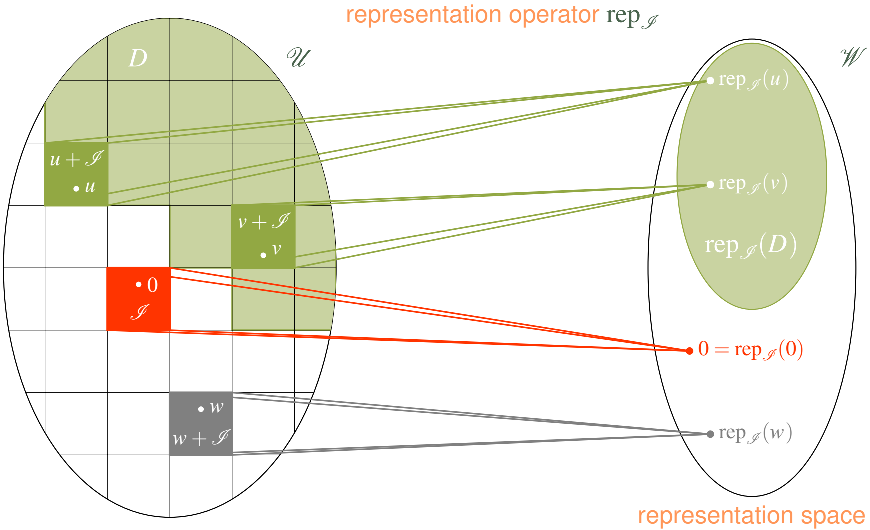

Indifference: representation

Representation

A representation for $\mathscr{I}$ consists of a representation space $\mathscr{W}$ and a representation operator $\operatorname{rep}_{\mathscr{I}} \colon \mathscr{U} \to \mathscr{W}$ such that

- $\mathscr{W}$ is a real linear space and $\operatorname{rep}_{\mathscr{I}}$ is a linear map;
- $\operatorname{rep}_{\mathscr{I}}$ is onto: $\operatorname{rng}(\operatorname{rep}_{\mathscr{I}}) = \mathscr{W}$;
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The indifference classes on the original space $\mathscr{U}$ are then given by:

$$[u]_{\mathscr{I}} = u + \mathscr{I} = \operatorname{rep}_{\mathscr{I}}^{-1}\big(\{\operatorname{rep}_{\mathscr{I}}(u)\}\big) = \big\{v \in \mathscr{U} \colon \operatorname{rep}_{\mathscr{I}}(v) = \operatorname{rep}_{\mathscr{I}}(u)\big\}.$$

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Do representations always exist?

Indifference: representation

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Inherited background ordering on $\mathscr{W}$

$$w \succ^{\star} 0 \Leftrightarrow (\exists u \in \mathscr{U})\big(w = \mathrm{rep}_{\mathscr{I}}(u) \text{ and } u \succ 0\big).$$

Indifference: representation

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- $\ker(\operatorname{rep}_{\mathscr{I}}) = \mathscr{I}$.

Representation theorem

A coherent set D of desirable options in $\mathscr{U}$ is $\mathscr{I}$-compatible if and only if there's some coherent set $D^\star$ of desirable options in $\mathscr{W}$ such that $D = \operatorname{rep}_{\mathscr{I}}^{-1}(D^\star) = \{u \colon \operatorname{rep}_{\mathscr{I}}(u) \in D^\star\}$, and this representation $D^\star$ is then uniquely given by $D^\star = \operatorname{rep}_{\mathscr{I}}(D) = \{\operatorname{rep}_{\mathscr{I}}(u) \colon u \in D\}$.

Indifference: representation

Representation

A representation for $\mathscr{I}$ consists of a representation space $\mathscr{W}$ and a representation operator $\operatorname{rep}_{\mathscr{I}} \colon \mathscr{U} \to \mathscr{W}$ such that

- $\mathscr{W}$ is a real linear space and $\operatorname{rep}_{\mathscr{I}}$ is a linear map;
- $\operatorname{rep}_{\mathscr{I}}$ is onto: $\operatorname{rng}(\operatorname{rep}_{\mathscr{I}}) = \mathscr{W}$;
- $\ker(\operatorname{rep}_{\mathscr{I}}) = \mathscr{I}$.

Representation theorem

Coherence and $\mathscr{I}$-compatibility on the original space are taken care of by mere coherence on the (simpler) representation space.

WHY BOTHER?

SYMMETRY

Symmetry

abstract	**gambles**
$\mathscr{U}$	$\mathscr{G}(\mathscr{X})$
u	f
$\succ$	weak ordering

Consider working with gambles f on an uncertain variable X in $\mathscr{X}$, so $f \in \mathscr{G}(\mathscr{X})$.

There is a symmetry behind X, modelled by a monoid $\mathscr{T}$ of transformations $T\colon \mathscr{X} \to \mathscr{X}$:

- $T_1 \circ T_2 \in \mathscr{T}$ for all $T_1, T_2 \in \mathscr{T}$;
- $\mathbf{1}_{\mathscr{T}} \circ T = T \circ \mathbf{1}_{\mathscr{T}} = T$ for all $T \in \mathscr{T}$.

Symmetry

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- $\mathbf{1}_{\mathscr{T}} \circ T = T \circ \mathbf{1}_{\mathscr{T}} = T$ for all $T \in \mathscr{T}$.

The effect of the symmetry assessment is indifference:

You are indifferent between any gamble f and any of its transforms $Tf := f \circ T$, so $f \equiv_{\mathscr{T}} Tf$.

This leads to a linear space of indifferent gambles

$$\mathscr{I}_{\mathscr{T}} := \operatorname{span}\big(\{f - Tf : f \in \mathscr{G}(\mathscr{X}) \text{ and } T \in \mathscr{T}\}\big).$$

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Consistency condition

There are coherent sets of desirable gambles on $\mathscr{X}$ that are $\mathscr{I}_{\mathscr{T}}$-compatible if and only if

$$\mathscr{I}_{\mathscr{T}} \cap \mathscr{G}(\mathscr{X})_{\succ 0} = \emptyset.$$

Symmetry

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Consequence:

$$\sup g \geq 0 \text{ for all } g \in \mathscr{I}_{\mathscr{T}}.$$

Symmetry

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Consequence: $\mathscr{T}$ is amenable!

$$\sup g \geq 0 \text{ for all } g \in \mathscr{I}_{\mathscr{T}}.$$

Necessary and sufficient condition for the existence of invariant coherent previsions P on $\mathscr{G}(\mathscr{X})$:

$$P(f) = P(Tf) \text{ for all gambles } f \in \mathscr{G}(\mathscr{X}) \text{ and all } T \in \mathscr{T}.$$

$\mathscr{M}_{\mathscr{T}}$ is the set of all such invariant coherent previsions.

Symmetry

Evaluation gambles

$\mathscr{G}^*(\mathscr{M}_{\mathscr{T}})$ is the linear space of all evaluation gambles

$$f^*\colon \mathscr{M}_{\mathscr{T}} \to \mathbb{R}\colon P \mapsto f^*(P) := P(f), \text{ for all gambles } f.$$

abstract	**gambles**
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Symmetry

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u	f
$\succ$	weak ordering
$\equiv$	$\equiv_{\mathscr{T}}$
$\mathscr{I}$	$\mathscr{I}_{\mathscr{T}}$
$\mathscr{W}$	$\mathscr{G}^*(\mathscr{M}_{\mathscr{T}})$
$\operatorname{rep}_{\mathscr{I}}$	$\operatorname{rep}_{\mathscr{I}_{\mathscr{T}}}$

Evaluation gambles

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Representation

Take as representation space $\mathscr{G}^*(\mathscr{M}_{\mathscr{T}})$ and as representation operator the onto map

$$\operatorname{rep}_{\mathscr{I}_{\mathscr{T}}}\colon \mathscr{G}(\mathscr{X}) \to \mathscr{G}^*(\mathscr{M}_{\mathscr{T}})\colon f \mapsto \operatorname{rep}_{\mathscr{I}_{\mathscr{T}}}(f) := f^*,$$

then, under some conditions,

$$\ker(\operatorname{rep}_{\mathscr{I}_{\mathscr{T}}}) = \mathscr{I}_{\mathscr{T}}!$$

CONDITIONING

Conditioning in probability theory

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$\mathscr{U}$	$\mathscr{G}(\mathscr{X})$
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Consider working with gambles f on an uncertain variable X in $\mathscr{X}$.

You start out with a coherent set of desirable gambles D, and then get the new information that the event $A \subset \mathscr{X}$ has occurred.

Conditioning in probability theory

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You start out with a coherent set of desirable gambles D, and then get the new information that the event $A \subset \mathscr{X}$ has occurred.

Two gambles f and g that have the same behaviour on A are now indifferent to You:

$$f \equiv_A g \Leftrightarrow \mathbb{I}_A f = \mathbb{I}_A g \text{ and } \mathscr{I}_A = \{h \in \mathscr{G}(\mathscr{X}) \colon \mathbb{I}_A h = 0\}.$$

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$\operatorname{rep}_{\mathscr{I}}$	$\operatorname{rep}_{\mathscr{I}_A}$
$\mathscr{W}$	$\mathscr{G}(A)$

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The representation operator $\operatorname{rep}_{\mathscr{I}_A}$ is in this case

$$\operatorname{rep}_{\mathscr{I}_A} : \mathscr{G}(\mathscr{X}) \to \mathscr{G}(A) : f \mapsto f|_A.$$

The representation space is now $\mathscr{G}(A)$.

Conditioning in probability theory

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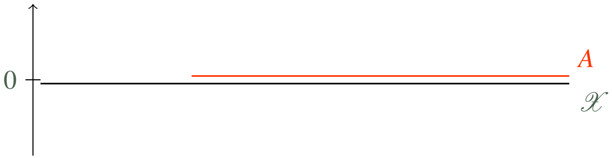
But there is a problem!

$$\mathscr{I}_A \cap \mathscr{G}(\mathscr{X})_{\succ 0} \neq \emptyset.$$

THERE ARE NO COHERENT AND $\mathscr{I}_A$-COMPATIBLE MODELS.



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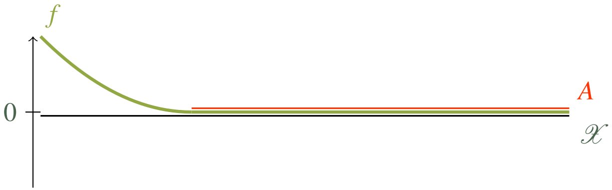

REPRESENTATION CAN NEVER WORK!

Conditioning in probability theory

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Interpretation to the rescue!

On the representing space A:

$$D\rfloor A := \{g \in \mathscr{G}(A) \colon g\mathbb{I}_A \in D\}$$ is coherent.

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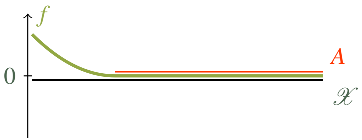

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On the representing space A:

$$D\rfloor A := \{g \in \mathscr{G}(A) \colon g\mathbb{I}_A \in D\} \text{ is coherent.}$$

On the original space $\mathscr{X}$:

$$D\|A := \{f \in \mathscr{G}(\mathscr{X}) \colon f\mathbb{I}_A \in D\}$$
$$= \operatorname{rep}^{-1}_{\mathscr{I}_A}(D\rfloor A) \text{ is } \mathscr{I}_A\text{-compatible but not coherent}$$

Conditioning in probability theory

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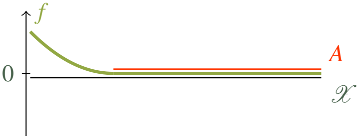

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$$\begin{aligned} D\|A &:= \{f \in \mathscr{G}(\mathscr{X}) \colon f\mathbb{I}_A \in D\} \\ &= \operatorname{rep}^{-1}_{\mathscr{I}_A}(D\rfloor A) \text{ is } \mathscr{I}_A\text{-compatible but not coherent} \end{aligned}$$

$$\begin{aligned} D|A &:= \operatorname{rep}^{-1}_{\mathscr{I}_A}(D\rfloor A) \cup \mathscr{G}(\mathscr{X})_{\succ 0} \\ &= \{f \in \mathscr{G}(\mathscr{X}) \colon f\mathbb{I}_A \in D \text{ or } f \succ 0\}. \end{aligned}$$

$D|A$ is as close as we can get to $\mathscr{I}_A$-compatibility while maintaining coherence.

QUESTIONS?