The Set Structure of Precision

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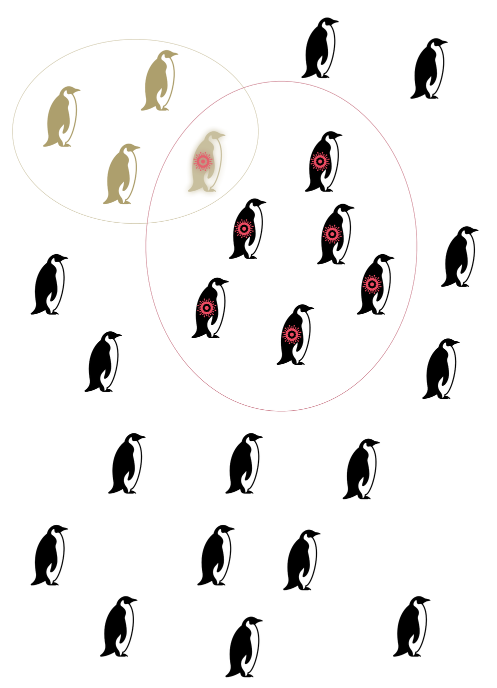
Modelling by Measurability — Motivation I

What do the following scenarios have in common?

- Machine learning algorithm with restricted access: privacy preservation, "not-missing-at-random" features, accelerated database access, multi-measurement data [1].
- Incompatible Quantum physical quantities, e.g. location and impulse [2].
- Preference ordering gives precise beliefs on events not closed under intersections [3].
- Frequential probability [4].



(Pre-)Dynkin-System as domain of probability.

Probability on Pre-Dynkin-System — Definitions

A **pre-Dynkin-system** $\mathcal{D} \subseteq 2^\Omega$ on Ω fulfills:

- $\emptyset \in \mathcal{D}$,
- $D \in \mathcal{D}$ implies $D^c := \Omega \setminus D \in \mathcal{D}$,
- $C, D \in \mathcal{D}$ with $C \cap D = \emptyset$ implies $C \cup D \in \mathcal{D}$.

A function $\mu \colon \mathcal{D} \to [0,1]$ is a (finitely additive) **probability on** $\mathcal{D}$, if:

- $\mu(\emptyset) = 0$ and $\mu(\Omega) = 1$.
- Let $C, D \in \mathcal{D}$ such that $C \cap D = \emptyset$, then $\mu(C \cup D) = \mu(C) + \mu(D)$.

Extendability = Coherence — Linking together

- A probability $\mu \colon \mathcal{D} \to [0,1]$ is **extendable** if there exists a probability $\mu' \colon 2^\Omega \to [0,1]$ such that $\mu'|_\mathcal{D} = \mu$.
- A probability on a pre-Dynkin-system is extendable if and only if it is coherent [5,6].
- There exists an inner and outer measure extension (μ_*, μ^*) for every probability, and a coherent extension $(\underline{\mu}_\mathcal{D}, \overline{\mu}_\mathcal{D})$ for every extendable probability.

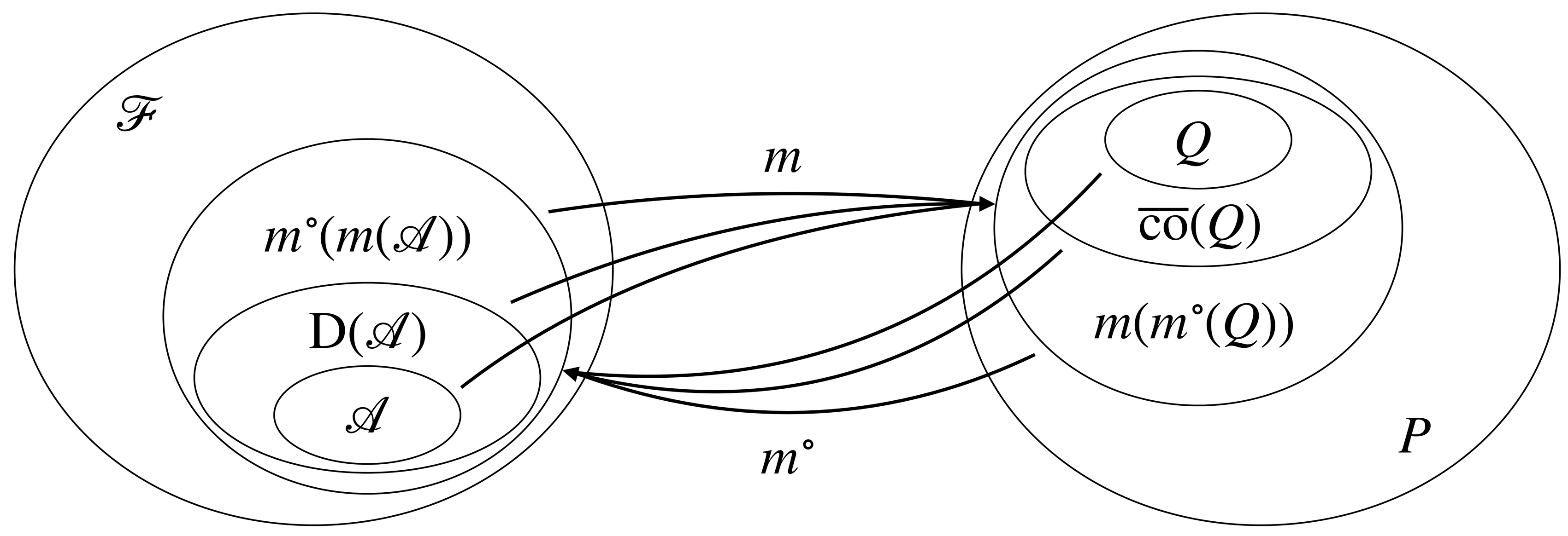
(Dual) Credal Set Function — Set System to IP

Let $\mathscr{F}$ be an algebra on Ω and P all probabilities on $\mathscr{F}$. For a fixed finitely additive probability $\psi \in P$, we define the **credal set function**,

$$m \colon 2^\mathscr{F} \to 2^P, m(\mathscr{A}) := \{\nu \in P \colon \nu(A) = \psi(A),\ \forall A \in \mathscr{A}\},$$

and the **dual credal set function**,

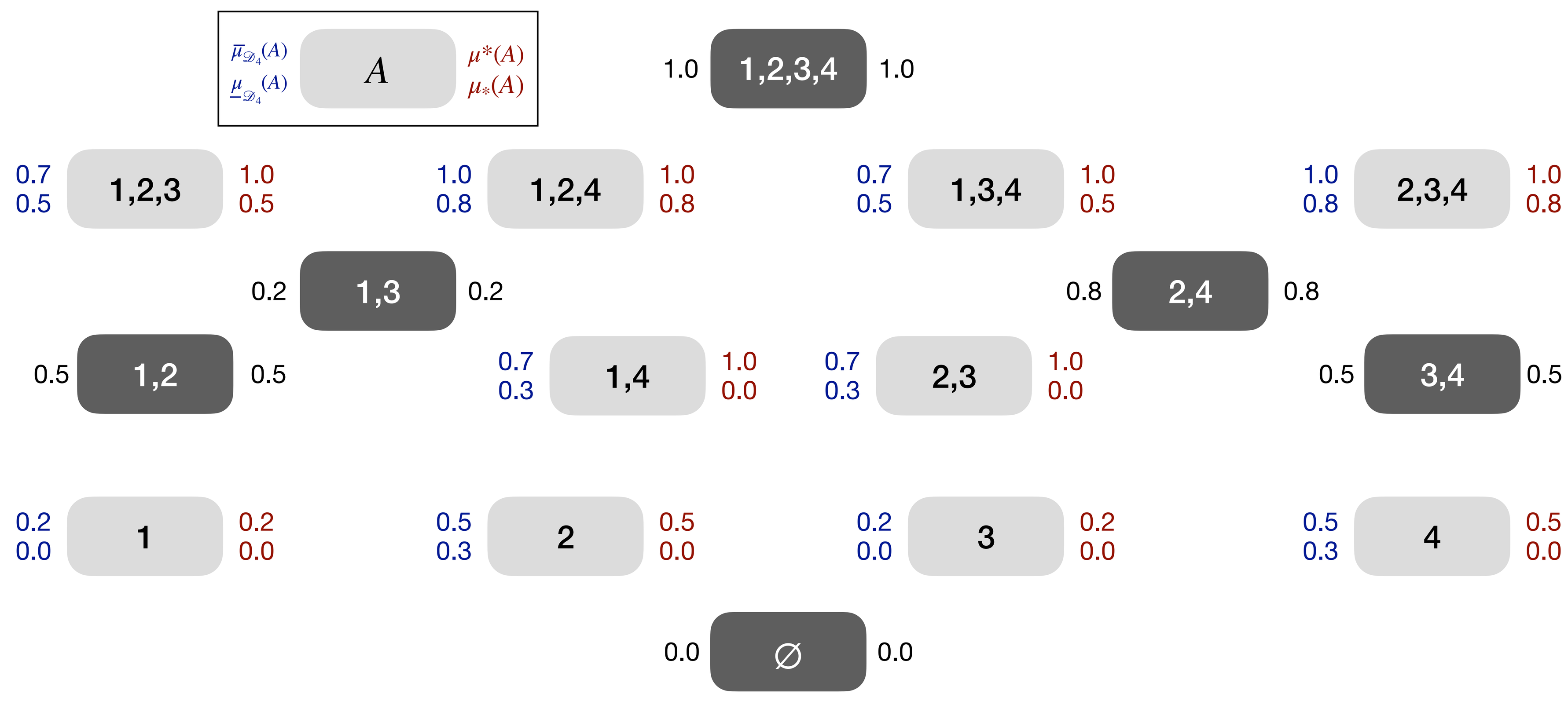
$$m^\circ \colon 2^P \to 2^\mathscr{F}, m^\circ(\mathcal{Q}) := \{A \in \mathscr{F} \colon \nu(A) = \psi(A), \forall \nu \in \mathcal{Q}\}.$$



We denote the pre-Dynkin-hull with D and the convex closure with $\overline{\mathrm{co}}$.

System of Precision — Motivation II

Imprecise Probability (IP) focuses on imprecision.
We ask: **On which events are imprecise probabilities precise?**



The Set Structure of Precision — Simple but Insightful

Let $\mathscr{F}$ be an algebra on Ω. Let $\ell \colon \mathscr{F} \to [0,1]$ and $u \colon \mathscr{F} \to [0,1]$ be two set functions s.t.:

Normalization: $u(\emptyset) = \ell(\emptyset) = 0$.
Conjugacy: $u(A) = 1 - \ell(A^c)$ for $A, A^c \in \mathscr{F}$.
Subadditivity of u**:** for $A, B \in \mathscr{F}$ such that $A \cap B = \emptyset$ then $u(A \cup B) \leq u(A) + u(B)$.
Superadditivity of ℓ**:** for $A, B \in \mathscr{F}$ such that $A \cap B = \emptyset$ then $\ell(A \cup B) \geq \ell(A) + \ell(B)$.

Then u and ℓ define a finitely additive probability measure $\mu := u|_\mathcal{D} = \ell|_\mathcal{D}$ on a pre-Dynkin-system $\mathcal{D} \subseteq \mathscr{F}$.

Coherent imprecise probabilities [5] fulfill all the required properties.

Set Systems ↔ IP — A Lattice Duality

- The set $m(\mathcal{Q})$ is a credal set, i.e. non-empty, weak*-closed and convex subset of P.
- The set $m^\circ(\mathscr{A})$ is a pre-Dynkin-system.
- The credal set function m and the dual credal set function m° form a so-called Galois connection, i.e. $\mathscr{A} \subseteq m^\circ(\mathcal{Q}) \Leftrightarrow \mathcal{Q} \subseteq m(\mathscr{A})$.

Galois connections induce a lattice duality between so-called bipolar-closed set systems and credal sets.

Departing From Here — Future

- Generalize from probabilities and set structures to expectation-type functional and systems of gambles.
- Coherent imprecise probabilities parametrized by pre-Dynkin-systems versus other parametrizations.
- Social intersectionality and pre-Dynkin-systems.

The FMLS group — Foundations of Machine Learning Systems

Goal: to develop new and better foundations for machine learning (ML) systems

References

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