# Sublinear Expectations ...

For a domain $\mathcal{D} \subseteq \mathbb{R}^{\Omega}$ which includes all constant functions, a sublinear/linear expectation on $\mathcal{D}$ is a functional $\overline{E}/E\colon \mathcal{D} \to \mathbb{R}$ that is constant preserving, isotone and ...

... sublinear, meaning that

$$\overline{E}(\alpha f + g) \leq \alpha \overline{E}(f) + \overline{E}(g)$$

for all $f, g \in \mathcal{D}$ and $\alpha \in \mathbb{R}_{\geq 0}$ with $\alpha f + g \in \mathcal{D}$.

... linear, meaning that

$$E(\alpha f + g) = \alpha E(f) + E(g)$$

for all $f, g \in \mathcal{D}$ and $\alpha \in \mathbb{R}$ with $\alpha f + g \in \mathcal{D}$.

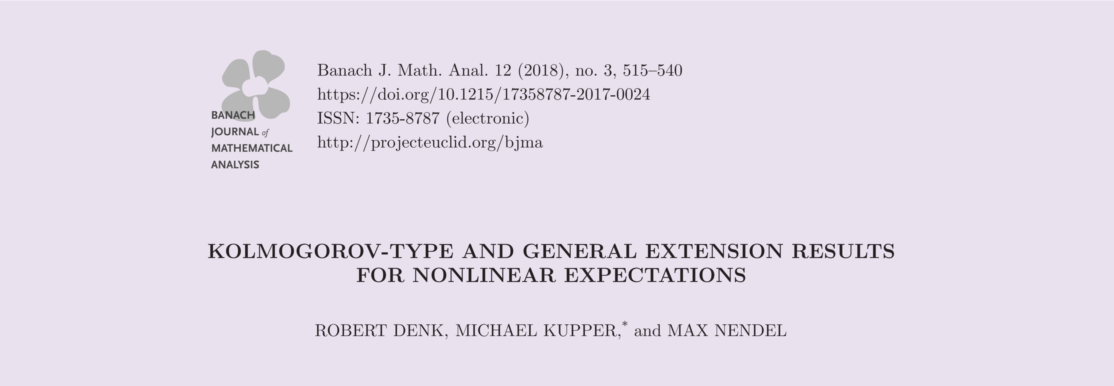
Such a sublinear expectation $\overline{E}$ is said to be **downward continuous** on $\mathcal{S} \subseteq \mathcal{D}$ if

$$\lim_{n\to+\infty} \overline{E}(f_n) = \overline{E}(f) \text{ for all } \mathcal{S}^{\mathbb{N}} \ni (f_n)_{n\in\mathbb{N}} \searrow f \in \mathcal{S}$$

and **upward continuous** on $\mathcal{S} \subseteq \mathcal{D}$ if

$$\lim_{n\to+\infty} \overline{E}(f_n) = \overline{E}(f) \text{ for all } \mathcal{S}^{\mathbb{N}} \ni (f_n)_{n\in\mathbb{N}} \nearrow f \in \mathcal{S}.$$

straightforward modification



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KOLMOGOROV-TYPE AND GENERAL EXTENSION RESULTS FOR NONLINEAR EXPECTATIONS

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Suppose $\mathcal{D} \subseteq \mathcal{L}(\Omega)$ is a linear lattice.

$\overline{E}$ is downward (& then upward) continuous on $\mathcal{D}$ iff every dominated linear expectation in

$$\mathbb{E}_{\overline{E}} := \left\{E \in \mathbb{E}(\mathcal{D})\colon (\forall f \in \mathcal{D})\ E(f) \leq \overline{E}(f)\right\}$$

is downward continuous.

$E$ is downward (& then upward) continuous on $\mathcal{D}$ iff there is a unique probability measure $P_E$ on $\sigma(\mathcal{D})$ such that

$$E(f) = \int f \,\mathrm{d}P_E \quad \text{for all } f \in \mathcal{D}.$$

also for convex $\overline{E}$

**Theorem**

The sublinear expectation $\overline{E}^{\sigma}$ extends $\overline{E}$, is downward continuous on $\mathcal{D}_{\delta} \cap \mathcal{L}(\Omega)$ and upward continuous on $\mathcal{M}_{\mathrm{b}}(\mathcal{D})$.

On $\mathcal{M}_{\mathrm{b}}(\mathcal{D})$, this extension is unique.

Let $\mathcal{M}(\mathcal{D}) := \mathcal{M}_{\mathrm{b}}(\mathcal{D}) \cup \mathcal{M}^{\mathrm{b}}(\mathcal{D})$ be the set of $\sigma(\mathcal{D})$-measurable variables $f \in \mathbb{R}^{\Omega}$ that are bounded below/above and let

$$\overline{E}^{\sigma}\colon \mathcal{M}(\mathcal{D}) \to \overline{\mathbb{R}}\colon f \mapsto \sup\left\{\int f \,\mathrm{d}P_E\colon E \in \mathbb{E}_{\overline{E}}\right\}.$$

# ... for Countable-State Uncertain Processes

Let $\mathscr{X}$ denote the countable state space. The possibility space $\Omega$ is **some** set of *paths* $\omega\colon \mathbb{R}_{\geq 0} \to \mathscr{X}$, and the domain $\mathcal{D}$ are the **finitary bounded variables**:

$$\mathcal{D} := \left\{g(X_{t_1}, \ldots, X_{t_n})\colon n \in \mathbb{N}, t_1 < \cdots < t_n \in \mathbb{R}_{\geq 0}, g \in \mathcal{L}(\mathscr{X}^n)\right\} \quad \text{with } X_t\colon \Omega \to \mathscr{X}\colon \omega \mapsto \omega(t).$$

sublinear expectation $\overline{E}_0$ on $\mathcal{L}(\mathscr{X})$

¿sublinear process $\overline{E}$ on $\mathcal{D}$?

semigroup $(\overline{\mathrm{T}}_t\colon \mathcal{L}(\mathscr{X}) \to \mathcal{L}(\mathscr{X}))_{t\in\mathbb{R}_{\geq 0}}$ of 'sublinear transition operators':

(i) $\overline{\mathrm{T}}_t[\bullet](x)$ is a sublinear expectation
(ii) $\overline{\mathrm{T}}_0 = \mathrm{I}$
(iii) $\overline{\mathrm{T}}_{s+t} = \overline{\mathrm{T}}_s \circ \overline{\mathrm{T}}_t$

¡sublinear Markov process!

$$(\forall n \in \mathbb{N}; t_1 < \cdots t_n \in \mathbb{R}_{\geq 0}; x_1, \ldots, x_n \in \mathscr{X})(\exists \omega \in \Omega)\ \omega(t_1) = x_1, \ldots, \omega(t_n) = x_n$$

**Theorem**

There is a unique sublinear expectation $\overline{E}$ on $\mathcal{D}$ such that

(i) $\overline{E}(g(X_0)) = \overline{E}_0(g)$ for all $g \in \mathcal{L}(\mathscr{X})$ and
(ii) for all $s_1 < \cdots < s_n < t \in \mathbb{R}_{\geq 0}$ and $g \in \mathcal{L}(\mathscr{X}^{n+1})$,

$$\overline{E}(g(X_{s_1}, \ldots, X_{s_n}, X_t)) = \overline{E}(h(X_{s_1}, \ldots, X_{s_n}))$$

with $h \in \mathcal{L}(\mathscr{X}^{\{s_1,\ldots,s_n\}})$ defined by

$$h(x_{s_1}, \ldots, x_{s_n}) := \overline{\mathrm{T}}_{t-s_n}\left[g(x_{s_1}, \ldots, x_{s_n}, \bullet)\right](x_{s_n}).$$

Is this corresponding $\overline{E}$ downward continuous on $\mathcal{D}$?

A semigroup $(\overline{\mathrm{T}}_t)_{t\in\mathbb{R}_{\geq 0}}$ of sublinear transition operators ...

... has **uniformly bounded rate** if

$$\limsup_{t\searrow 0} \frac{1}{t} \sup\left\{\overline{\mathrm{T}}_t[1 - \mathbb{I}_x](x)\colon x \in \mathscr{X}\right\} < +\infty,$$

or NEW equivalently, $\limsup_{t\searrow 0} \frac{1}{t}\|\overline{\mathrm{T}}_t - \mathrm{I}\| < +\infty$ NEW.

NEW

... is **uniformly continuous** if

$$\lim_{t\searrow 0} \|\overline{\mathrm{T}}_t - \mathrm{I}\| = 0.$$

$\Omega := \mathrm{cdlg}(\mathscr{X}^{\mathbb{R}_{\geq 0}}) \subsetneq \mathscr{X}^{\mathbb{R}_{\geq 0}}$

$\overline{E}_0$ is downward continuous
&
$\overline{\mathrm{T}}_t[\bullet](x)$ is downward continuous
&
$(\overline{\mathrm{T}}_t)_{t\in\mathbb{R}_{\geq 0}}$ has uniformly bounded rate

**$\overline{E}$ is downward continuous on $\mathcal{D}$**

***$\mathcal{M}(\mathcal{D})$ is sufficiently rich***

$\Omega := \mathscr{X}^{\mathbb{R}_{\geq 0}}$

$\overline{E}_0$ is downward continuous
&
$\overline{\mathrm{T}}_t[\bullet](x)$ is downward continuous

**$\overline{E}$ is downward continuous on $\mathcal{D}$**

***$\mathcal{M}(\mathcal{D})$ is not sufficiently rich***

NEW

For some 'bounded sublinear rate operator' $\overline{\mathrm{Q}}\colon \mathcal{L}(\mathscr{X}) \to \mathcal{L}(\mathscr{X})$,

$$\overline{\mathrm{T}}_t = e^{t\overline{\mathrm{Q}}} := \lim_{n\to+\infty} \left(\mathrm{I} + \frac{t}{n}\overline{\mathrm{Q}}\right)^n \text{ for all } t \in \mathbb{R}_{\geq 0};$$

whenever this is the case,

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{\mathrm{T}}_t := \lim_{s\to t} \frac{\overline{\mathrm{T}}_s - \overline{\mathrm{T}}_t}{|s - t|} = \overline{\mathrm{Q}}\,\overline{\mathrm{T}}_t \text{ for all } t \in \mathbb{R}_{\geq 0}.$$

## sublinear Poisson process

Fix some rate interval $[\underline{\lambda}, \overline{\lambda}] \subset \mathbb{R}_{\geq 0}$, and take

$$\mathscr{X} := \mathbb{Z}_{\geq 0}, \quad \overline{E}_0(g) := g(0) \quad \& \quad (\overline{\mathrm{T}}_t)_{t\in\mathbb{R}_{\geq 0}} := (e^{t\overline{\mathrm{L}}})_{t\in\mathbb{R}_{\geq 0}},$$

where $\overline{\mathrm{L}}\colon \mathcal{L}(\mathscr{X}) \to \mathcal{L}(\mathscr{X})$ maps $g \in \mathcal{L}(\mathscr{X})$ to

$$\mathscr{X} \to \mathbb{R}\colon x \mapsto \max\left\{\lambda\left(g(x+1) - g(x)\right)\colon \lambda \in [\underline{\lambda}, \overline{\lambda}]\right\}.$$

Cool, on $\mathcal{M}_{\mathrm{b}}(\mathcal{D})$ there is a ...... 'sufficiently continuous' extension of the downward continuous sublinear expectation $\overline{E}$ on $\mathcal{D}$!

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