A Comparison Between a Frequentist, Bayesian and Imprecise Bayesian Approach to Delay Time Maintenance

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1. Introduction

Ultimate goal: optimise the inspection period T [1].

$Y = U + H$, with Y : **failure time**, U : **defect appearance time**, H : **delay time**.

$$f_Y(y) = k_f \lambda_h e^{-k_f y} \int_{h=0}^{y} e^{(k_f - \lambda_h)h} dh = \frac{k_f \lambda_h e^{-k_f y}}{k_f - \lambda_h}\left(e^{(k_f-\lambda_h)y} - 1\right) \text{ when } \lambda_h \neq k_f.$$

We seek to minimise $D_1(T) = \frac{E(Downtime)}{E(Operating\ Time)}$ and $C_1(T) = \frac{E(Cost)}{E(Operating\ Time)}$.

$$E(Downtime) = p_{intact} d_i + p_{defect}(d_i + d_{defect}) + p_{failure} d_{br}$$

$$E(Operating\ Time) = E(Y|Y \leq T) F_Y(T) + T\left(1 - F_Y(T)\right)$$

$$E(Cost) = p_{intact} c_i + p_{defect}(c_i + c_{defect}) + p_{failure} c_{br}$$

2. Virtual case study

Pump where one defect progressively deteriorates into a failure.

k_f = 1/4 defects/month, λ_h = 9/100 failures/defect/month

However, the enterprise only knows that $\lambda_h \in [0.01; 0.1]$ failures/defect/month.

d_i = 0.4 days, d_{defect} = 2 days, d_{br} = 28 days, c_i = 110 €, c_{defect} = 1000 €, and c_{br} = 9000 €.

8 samples of failure times $S_1 \subset S_2 \subset S_3 \subset S_4 \subset S_5 \subset S_6 \subset S_7 \subset S_8$.

Sizes of the samples : 2, 5, 10, 20, 60, 200, 600, and 1000 .

Let $T_{D_1,opt,\lambda_{h,true}}$ be the optimal inspection period minimising $D_1(T)$ knowing $\lambda_{h,true}$

The enterprise will use this type of pumps in **6** factories for a period equal to t = **4 years** of operational time.

Consequently, it is interesting to consider the quantities

$$\Delta Downtime = 6\left(D_1(T_{D_1,opt,S}, \lambda_{h,true}) - D_1(T_{D_1,opt,\lambda_{h,true}}, \lambda_{h,true})\right)t$$

$$\Delta Cost = 6\left(C_1(T_{C_1,opt,S}, \lambda_{h,true}) - C_1(T_{C_1,opt,\lambda_{h,true}}, \lambda_{h,true})\right)t$$

Suboptimality of the decision based on sample S.

3. Frequentist approach

Maximise the likelihood $L(\lambda_h|y_1, y_2, \ldots, y_n) = \prod_{i=1}^{n} f_Y(y_i) = \prod_{i=1}^{n} \frac{k_{f,0}\lambda_h}{k_{f,0} - \lambda_h}\left(e^{-\lambda_h y_i} - e^{-k_{f,0} y_i}\right)$.

Sample Size	$\lambda_{h,MLE}$	$T_{D_1,opt,S}$	$T_{C_1,opt,S}$	Δ Downtime	Δ Cost (€)
2	0.0662	60.3735	67.7731	3.2699	1428.5680
5	0.0690	58.3667	64.6834	2.4081	1033.6236
10	0.0960	45.6374	47.2540	0.1278	49.4582
20	0.0946	46.1062	47.8410	0.0766	29.7284
60	0.0860	49.3381	51.9885	0.0654	26.0579
200	0.0862	49.2551	51.8798	0.0589	23.4314
600	0.0880	48.5267	50.9301	0.0159	6.2836
1000	0.0902	47.6784	49.8361	0.0002	0.0605
$+\infty$	0.0900	47.7537	49.9327	0.0000	0.0000

Poor decisions for <u>small</u> sample sizes !

Literature

[1] RD Baker and W Wang. Estimating the delay-time distribution of faults in repairable machinery from failure data. IMA Journal of Management Mathematics, 3(4): 259–281, 1991.

[2] Christian P Robert et al. The Bayesian choice: from decision-theoretic foundations to computational implementation, volume 2. Springer, 2007.

[3] Seamus Bradley. Imprecise Probabilities. In Edward N. Zalta, editor, The Stanford Encyclopedia of Philosophy. Metaphysics Research Lab, Stanford University, Spring 2019 edition, 2019.

[4] W Wang and X Jia. An empirical Bayesian based approach to delay time inspection model parameters estimation using both subjective and objective data. Quality and Reliability Engineering International, 23(1):95–105, 2007.

4. Precise Bayesian approach

$$f(\lambda_h|S) = \frac{L(S|\lambda_h) f_{1,0}(\lambda_h)}{\int_{\lambda_h=0.01}^{0.1} L(S|\lambda_h) f_{1,0}(\lambda_h) d\lambda_h} \ , \ E(\lambda_h|S) = \int_{\lambda_h=0.01}^{0.1} \lambda_h f(\lambda_h|S)$$

Uniform prior

Sample Size	$E(\lambda_h\|S)$	$T_{D_1,opt,S}$	$T_{C_1,opt,S}$
0	0.0550	72.1056	88.7999
2	0.0613	64.9950	75.5063
5	0.0664	60.5123	68.0865
10	0.0785	52.9380	56.8024
20	0.0829	50.7739	53.8654
60	0.0843	50.0010	52.9380
200	0.0863	49.2281	51.8559
600	0.0881	48.4552	50.9285
1000	0.0903	47.6823	49.8464
$+\infty$	0.0900	47.7537	49.9327

Comparison with the frequentist approach

n_Y	$\Delta Downtime_B$	$\Delta Downtime_F$	$\Delta Cost_B$	$\Delta Cost_F$
2	5.5803	3.2699	2568.2310	1428.5680
5	3.3308	2.4081	1469.9209	1033.6236
10	0.6447	0.1278	261.8602	49.4582
20	0.2298	0.0766	91.3934	29.7284
60	0.1295	0.0654	54.4745	26.0579
200	0.0568	0.0589	22.8585	23.4314
600	0.0131	0.0159	6.2591	6.2836
1000	0.0001	0.0002	0.0481	0.0605
$+\infty$	0.0000	0.0000	0.0000	0.0000

$E_{\lambda_h}\left(D_1(T,\lambda_h)\right) = \int_{\lambda_h=0.01}^{0.1} D_1(T,\lambda_h) f(\lambda_h|S) d\lambda_h$ and $E_{\lambda_h}\left(C_1(T,\lambda_h)\right) = \int_{\lambda_h=0.01}^{0.1} C_1(T,\lambda_h) f(\lambda_h|S) d\lambda_h$

are minimised with respect to T [2].

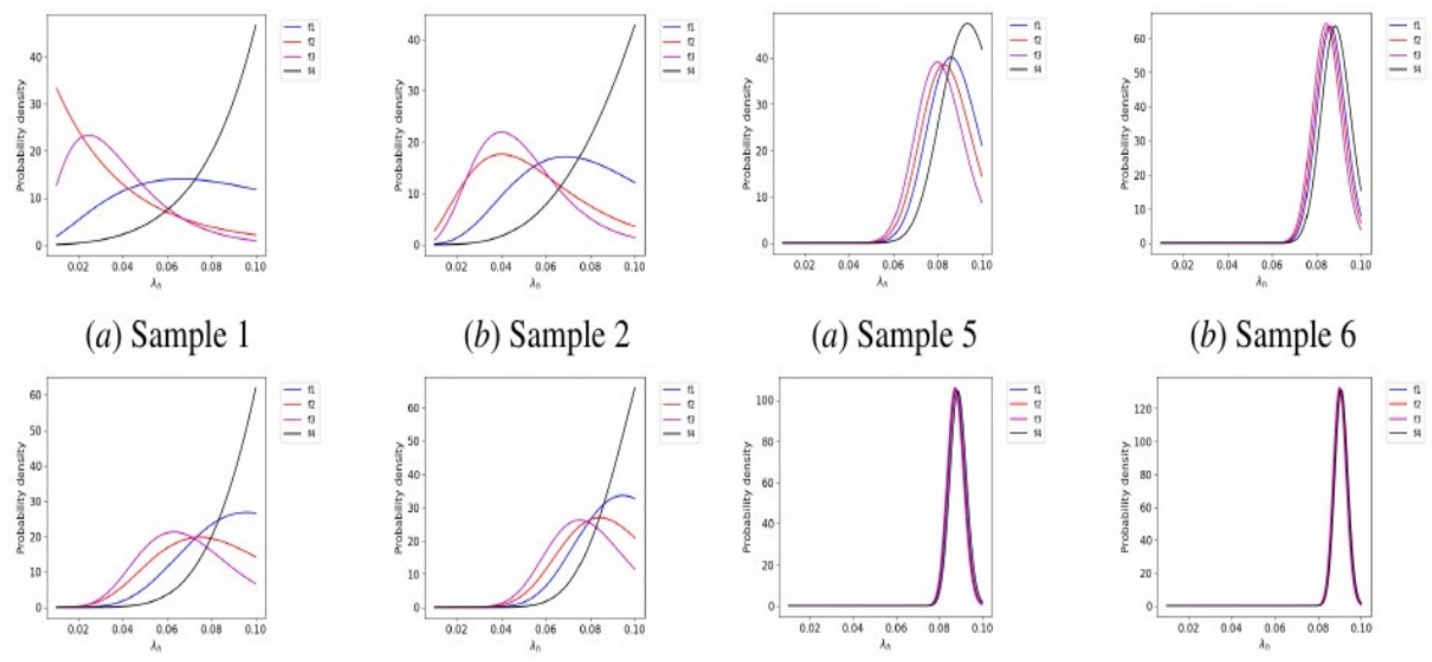
5. Imprecise Bayesian approach

$f_{1,0}(\lambda_h)$: prior <u>uniform</u> with respect to $\lambda_h \in [0.01; 0.1]$

$f_{2,0}(\lambda_h) = \frac{1}{\lambda_h^2} \frac{1}{\frac{1}{\lambda_{h,min}} - \frac{1}{\lambda_{h,max}}}$: prior <u>uniform</u> with respect to $E(H) = \frac{1}{\lambda_h} \in [10; 100]$

$f_{3,0}(\lambda_h) = \frac{te^{-\lambda_h t}}{e^{-0.01t} - e^{-0.1t}}$: prior <u>uniform</u> with respect to $prop_{Safe}(t,\lambda_h) \in [e^{-0.1t}, e^{-0.01t}]$

$f_{4,0}(\lambda_h) = t\frac{e^{\lambda_h t}}{e^{0.1t} - e^{0.01t}}$: prior <u>uniform</u> with respect to $\frac{1}{prop_{safe}(t,\lambda_h)} \in [e^{0.01t}; e^{0.1t}]$.



(*a*) Sample 1 (*b*) Sample 2 (*a*) Sample 5 (*b*) Sample 6

(*c*) Sample 3 (*d*) Sample 4 (*c*) Sample 7 (*d*) Sample 8

Posteriors

n_Y	$min(T_{D_1,opt})$	$max(T_{D_1,opt})$	$\Delta Downtime_B$	$\Delta Downtime_F$
0	51.7014	$+\infty$	9.8058	
2	51.5468	118.6333	5.5803	3.2699
5	51.3922	83.0805	3.3308	2.4081
10	49.0735	61.1306	0.6447	0.1278
20	48.3007	54.9475	0.2298	0.0766
60	48.6098	52.0105	0.1295	0.0654
200	48.4552	50.0010	0.0568	0.0589
600	48.1461	48.7644	0.0131	0.0159
1000	47.5278	47.8369	0.0001	0.0002

n_Y	$min(T_{C_1,opt})$	$max(T_{C_1,opt})$	$\Delta Cost_B$	$\Delta Cost_F$
0	55.1021	$+\infty$	4837.4707	
2	54.9475	$+\infty$	2568.2310	1428.5680
5	54.6383	114.7689	1469.9209	1033.6236
10	51.5468	69.0140	261.8602	49.4582
20	50.7739	59.5848	91.3934	29.7284
60	50.9285	55.5658	54.4745	26.0579
200	50.9285	52.7834	22.8585	23.4314
600	50.4647	51.2376	6.2591	6.2836
1000	49.5373	50.0010	0.0481	0.0605

Imprecise decision intervals

6. Conclusion

Both frequentism and precise Bayesianism fare poorly for small samples.

Precise Bayesians can only consider this imprecision through *ad-hoc* reasoning.

Imprecise Bayesians can readily capture the difference between ignorance and knowledge through probability intervals [3].

This approach should be extended to more complex DTM models [1].