Describing and Quantifying Contradiction between Pieces of Evidence

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Abstract

Belnap Dunn logic (BD) is a four-valued logic introduced to model reasoning with incomplete or contradictory information. In this article, we show how Dempster-Shafer (DS) theory can be used over BD in order to formalise reasoning with incomplete and/or contradictory pieces of evidence. First, we discuss how to encode different kinds of evidence, and how to interpret the resulting belief and plausibility functions. Then, we discuss the behaviour of Dempster's rule in this framework and present a variation of the rule. Finally, we show how to construct credal sets of classical probability measures based on this kind of evidence.

Keywords. Dempster-Shafer theory, Belnap Dunn logic, contradictory evidence.

Belief functions

Let $\mathcal{L}$ be a bounded lattice. A function $\texttt{bel} : \mathcal{L} \to [0,1]$ is a **belief function** if : (1) $\texttt{bel}(\bot) = 0$ and $\texttt{bel}(\top) = 1$, (2) $\texttt{bel}$ is *monotone*, and (3) for all $k \geq 1$ and all $a_1, \ldots, a_k \in \mathcal{L}$, we have

$$\texttt{bel}\left(\bigvee_{1\leq i\leq k} a_i\right) \geq \sum_{J\subseteq\{1,\ldots,k\} \text{ and } J\neq\varnothing} (-1)^{|J|+1} \cdot \texttt{bel}\left(\bigwedge_{j\in J} a_j\right).$$

A **mass fonction** on $\mathcal{L}$ is a fonction $\texttt{m} : \mathcal{L} \to [0,1]$ s.t. $\sum_{x\in\mathcal{L}} \texttt{m}(x) = 1$. Every belief function $\texttt{bel}$ on a finite lattice can be represented via a mass function $\texttt{m}_{\texttt{bel}}$ and vice-versa. We have $\texttt{bel}(x) = \sum_{y\leq x} \texttt{m}_{\texttt{bel}}(y)$.

Dempster-Shafer theory of evidence. Belief functions and their mass functions are used to reason on the available information. Dempster-Shafer combination rule allows the combining of pieces of evidence provided by different sources. Each source is described by a mass function.

Let $\texttt{m}_1$ and $\texttt{m}_2$ be two mass functions on $\mathcal{P}(S)$. The result of **Dempster–Shafer combination rule** $\texttt{m}_1 \oplus \texttt{m}_2 : \mathcal{P}(S) \to [0,1]$ is: $\texttt{m}_1 \oplus \texttt{m}_2(X) = 0$ if $X = \emptyset$, otherwise, we have $\texttt{m}_1 \oplus \texttt{m}_2(X)$ equal to

$$\frac{\sum\{\texttt{m}_1(X_1)\cdot\texttt{m}_2(X_2) \mid X_1\cap X_2 = X\}}{\sum\{\texttt{m}_1(X_1)\cdot\texttt{m}_2(X_2) \mid X_1\cap X_2 \neq \varnothing\}}.$$

Conflict in DS theory.

In Dempster's original combination rule, it is assumed that the sources are completely reliable, and hence any conflict between them is considered impossible. In addition, it is assumed that the frame of discernment Ω is composed of a list of mutually incompatible and exhaustive events. That is, every possible outcome is listed in Ω and no two outcomes in Ω can take place at the same time.

Zadeh gives an example to show that DS-*rule can lead to counterintuitive results when it is used to aggregate pieces of evidence that are not fully reliable and with a significant degree of conflict between them.* Several modifications of DS-rule have been proposed and studied in the literature to aggregate pieces of evidence both from not fully reliable sources and from sources strongly contradicting each other.

In this work, we use an expansion of Belnap Dunn logic (BD) to represent and combine conflicting evidence. BD was proposed to represent and reason about incomplete and contradictory information.

BD logic: Reasoning with incomplete and contradictory information

Intuitive interpretation. In BD, a statement p is either "supported by the information", or "refuted by the information", or "neither supported nor refuted by the information", or "both supported and refuted by the information". These four truth values are respectively denoted $\mathbf{T}$ (*true*), $\mathbf{F}$ (*false*), $\mathbf{N}$ (*neither*), $\mathbf{B}$ (*both*).

Let $\texttt{Prop}$ be a finite set of proposional variables and $\mathsf{Lit} := \texttt{Prop} \cup \{\neg p \mid p \in \texttt{Prop}\}$.
The langage $\mathscr{L}_{\mathsf{BD}}$ is defined via as follows: $\phi \coloneqq p \in \texttt{Prop} \mid \neg\phi \mid \phi\wedge\phi \mid \phi\vee\phi \mid \bot \mid \top$.

A BD**-model** is a tuple $\mathfrak{M} = \langle W, v^+, v^-\rangle$ s.t. $W \neq \varnothing$ is a finite set of states and $v^+, v^- : \texttt{Prop} \to \mathcal{P}(W)$. The relations $\vDash^+$ and $\vDash^-$ are defined as follows:

$$
\begin{array}{cc}
w \vDash^+ \top \quad w \nvDash^+ \bot & w \nvDash^- \top \quad w \vDash^- \bot \\
w \vDash^+ p \text{ iff } w \in v^+(p) & w \vDash^- p \text{ iff } w \in v^-(p) \\
w \vDash^+ \neg\phi \text{ iff } w \vDash^- \phi & w \vDash^- \neg\phi \text{ iff } w \vDash^+ \phi \\
w \vDash^+ \phi\wedge\phi' \text{ iff } w \vDash^+ \phi \text{ and } w \vDash^+ \phi' & w \vDash^- \phi\wedge\phi' \text{ iff } w \vDash^- \phi \text{ or } w \vDash^- \phi' \\
w \vDash^+ \phi\vee\phi' \text{ iff } w \vDash^+ \phi \text{ or } w \vDash^+ \phi' & w \vDash^- \phi\vee\phi' \text{ iff } w \vDash^- \phi \text{ and } w \vDash^- \phi'
\end{array}
$$

In the article we use an equivalent semantics based on the De Morgan algebra 4*.*

Extensions of a formula.

$$
\begin{array}{ll}
|\phi|^+ = \{w \in W \mid w \vDash^+ \varphi\} & |\phi|^- = \{w \in W \mid w \vDash^- \varphi\} \\
|\phi|^{\mathbf{T}} = \{w \in W \mid w \vDash^+ \varphi \text{ and } w \nvDash^- \varphi\} & |\phi|^{\mathbf{F}} = \{w \in W \mid w \nvDash^+ \varphi \text{ and } w \vDash^- \varphi\}
\end{array}
$$

Lindenbaum algebra. Let $\cong$ be the congruence relation on $\mathscr{L}_{\mathsf{BD}}$ defined as $\phi \cong \phi'$ iff $\phi \vDash_{\mathsf{BD}} \phi'$ and $\phi' \vDash_{\mathsf{BD}} \phi$. The BD *Lindenbaum algebra* over $\mathscr{L}_{\mathsf{BD}}$ is the De Morgan algebra $\langle\{[\phi]_{\phi\in\mathscr{L}_{\mathsf{BD}}}\}, \wedge, \vee, \neg\rangle$, where $[\phi]$ is the equivalence class of the formula ϕ, $\neg[\phi] = [\neg\phi]$ and $[\phi] \odot [\psi] = [\phi \odot \psi]$ for $\odot \in \{\wedge, \vee\}$. The CL *Lindenbaum algebra* over $\mathscr{L}_{\mathsf{BD}}$ is the Boolean algebra defined similarly using $\vDash_{\mathsf{CL}}$.

The canonical BD**-model over** $\texttt{Prop}$ is a tuple $\mathfrak{M}_c = \langle W_c, v^+, v^-\rangle$, where $W_c = \mathcal{P}(\mathsf{Lit})$ and the valuations $v^+, v^- : \texttt{Prop} \to W_c$ are defined as follows:

$$w \in v^+(p) \text{ iff } p \in w, \qquad w \in v^-(p) \text{ iff } \neg p \in w.$$

Probabilistic BD models

A *(paraconsistent) probability assignment* is a function $\mathrm{p} : \mathscr{L}_{\mathsf{BD}} \to [0,1]$ s.t., for all $\phi, \psi \in \mathscr{L}_{\mathsf{BD}}$, $\mathrm{p}(\bot) = 0$ and $\mathrm{p}(\top) = 1$, p is *monotone* (if $\phi \vDash_{\mathsf{BD}} \psi$, then $\mathrm{p}(\phi) \leq \mathrm{p}(\psi)$), and $\mathrm{p}(\phi\vee\psi) + \mathrm{p}(\phi\wedge\psi) = \mathrm{p}(\phi) + \mathrm{p}(\psi)$.

A *classical probability assignment* is a function $\mathrm{p} : \mathscr{L}_{CL} \to [0,1]$ s.t., for all $\phi, \psi \in \mathscr{L}_{CL}$, $\mathrm{p}(\top) = 1$, p is *monotone* (if $\phi \vDash_{\mathsf{CL}} \psi$, then $\mathrm{p}(\phi) \leq \mathrm{p}(\psi)$), and $\mathrm{p}(\phi\vee\psi) = \mathrm{p}(\phi) + \mathrm{p}(\psi)$, for $(\phi\wedge\psi) \vDash_{\mathsf{CL}} \bot$.

A *probabilistic* BD***-model*** is a tuple $\mathfrak{M} = \langle W, v^+, v^-, \mu\rangle$ such that $\langle W, v_4\rangle$ is a BD-model and $\mu : \mathcal{P}(W) \to [0,1]$ is a probability measure on $\mathcal{P}(W)$. The *induced probability assignment* is defined as follows: for any formula $\phi \in \mathscr{L}_{\mathsf{BD}}$,

$$\mathrm{p}(\phi) = \mu(|\phi|^+).$$

DS theory over BD

We consider a mass function $\texttt{m}$ on the canonical model $\mathfrak{M}_c = \langle W_c, v^+, v^-\rangle$. We note $\texttt{bel}_{\texttt{m}}$ the induced belief function $\texttt{bel}_{\texttt{m}}(X) = \sum_{Y\subseteq X} \texttt{m}(Y)$.

Encoding of evidence on the canonical model. The statement *"there is information supporting* p*"* is true at every state w s.t. $w \in v^+(p)$. Therefore, one encodes the statement *"the information 100% supports* p*"* via the mass function $\texttt{m}_p : \mathcal{P}(W_c) \to [0,1]$ such that $\texttt{m}(|p|^+) = 1$. The mass function $\texttt{m}_{\mathbf{T}p}$ such that $\texttt{m}_{\mathbf{T}p}(|p|^+ \cap (|p|^-)^c) = 1$ encodes classical evidence supporting p, i.e. *"the information 100% supports* p *and there is no information available supporting* $\neg p$*"*. We say that *"the information supports exactly* p*."*

Interpreting belief over BD**-models.** As the mass function is defined on the powerset algebra of the canonical model, $\texttt{bel}_{\texttt{m}}$ is a belief function and every combination rule using only the fact that the underlying algebra is a powerset algebra can still be used.

Lower and upper bounds on probability assignments. A mass function $\texttt{m}$ over W_c induces the following lower bound $\texttt{Bel}_{\mathbf{TB}}(\phi)$ on $\mathrm{p}(\phi)$:

$$\texttt{Bel}^+(\phi) \coloneqq \texttt{bel}_{\texttt{m}}(|\phi|^+), \qquad \texttt{Pl}^+(\phi) \coloneqq \texttt{pl}_{\texttt{m}}(|\phi|^+) = 1 - \texttt{bel}_{\texttt{m}}((|\phi|^+)^c).$$

Lower bounds for classical probabilities

What would be a 'good classical' piece of evidence for a statement ϕ in the paraconsistent framework, and therefore which notions of "belief" would be the most pertinent to estimate a lower bound on an unknown classical probability assignment p on the formulas.

- **Support based on classical 'proofs'.**
  Let $Q_w := \{l \in \mathsf{Lit} \mid w \in v^+(l) \text{ and } w \notin v^-(l)\}$. A state $w \in W$ supports ϕ if it provides a classical 'proof' of ϕ, i.e. if $\left(\bigwedge_{l\in Q_w} l\right) \vDash_{\mathsf{CL}} \phi$. Let $|\phi|^{\mathbf{CP}}$ be the set of states which provide a classical 'proof' of ϕ and

  $$\texttt{Bel}_{\mathbf{CP}}(\phi) = \texttt{bel}_{\texttt{m}}(|\phi|^{\mathbf{CP}}) \quad \text{and} \quad \texttt{Pl}_{\mathbf{CP}}(\phi) = 1 - \texttt{Bel}_{\mathbf{CP}}(|\neg\phi|^{\mathbf{CP}}).$$

- **Support from incomplete (resp. classical) states IC (resp. C).** We consider only incomplete, i.e., non-contradictory, (resp. classical, i.e., states corresponding to a classical valuation) states. Let $\mathbf{IC} \subseteq W_c$ (resp. $\mathbf{C} \subseteq W_c$), be the set of incomplete (resp. classical) states, and

  $$
  \begin{array}{ll}
  \texttt{Bel}_{\mathbf{IC}}(\phi) \coloneqq \texttt{bel}_{\texttt{m}}(|\phi|^{\mathbf{T}} \cap \mathbf{IC}), & \texttt{Pl}_{\mathbf{IC}}(\phi) = 1 - \texttt{bel}_{\texttt{m}}(|\phi|^{\mathbf{F}} \cap \mathbf{IC}) \\
  \texttt{Bel}_{\mathbf{C}}(\phi) \coloneqq \texttt{bel}_{\texttt{m}}(|\phi|^{\mathbf{T}} \cap \mathbf{C}) & \texttt{Pl}_{\mathbf{C}}(\phi) = 1 - \texttt{bel}_{\texttt{m}}(|\phi|^{\mathbf{F}} \cap \mathbf{C})
  \end{array}
  $$

Theorem. Let $\mathbb{B}$ be the CL Lindenbaum algebra over $\mathscr{L}_{\mathsf{BD}}$, CM be the set of classical probability measures on $\mathbb{B}$, $\mathbb{B}^* = \mathbb{B} \smallsetminus \{\top_{\mathbb{B}}\}$, and for $\mathcal{X} \in \{\mathbf{IC}, \mathbf{C}, \mathbf{CP}\}$,

$$\mathcal{F}_{\mathcal{X}} := \{\mu \in \mathrm{CM} \mid \forall a \in \mathbb{B},\ \texttt{bel}_{\mathcal{X}}(a) \leq \mu(a)\}.$$

For $\mathcal{X} \in \{\mathbf{IC}, \mathbf{C}\}$ (resp. $\mathcal{X} = \mathbf{CP}$), $\texttt{bel}_{\mathcal{X}}$ and $\texttt{pl}_{\mathcal{X}}$ are non-normal (resp. normal) belief and plausibility functions on $\mathbb{B}$, and, for $a \in \mathbb{B}^*$ (resp. $a \in \mathbb{B}$), $\texttt{bel}_{\mathcal{X}}(a)$ and $\texttt{pl}_{\mathcal{X}}(a)$ provide optimal lower and upper bounds on $\mathcal{F}_{\mathcal{X}}$. In addition,

$$\varnothing \subsetneq \mathcal{F}_{\mathbf{CP}} \subseteq \mathcal{F}_{\mathbf{IC}} \subseteq \mathcal{F}_{\mathbf{C}}.$$