Balanced Games

Michel GRABISCH

Université Paris I Panthéon-Sorbonne
Centre d'Economie de la Sorbonne
Paris School of Economics, Paris, France

- Cooperative games with transferable utility (*TU-games*) are merely set functions on a finite set vanishing on the empty set.

Introduction

- Cooperative games with transferable utility (*TU-games*) are merely set functions on a finite set vanishing on the empty set.
- This wide definition makes them appear in various fields of discrete mathematics: combinatorial optimization and operations research, game theory, decision theory, imprecise probabilities, etc.

Introduction

- Cooperative games with transferable utility (*TU-games*) are merely set functions on a finite set vanishing on the empty set.
- This wide definition makes them appear in various fields of discrete mathematics: combinatorial optimization and operations research, game theory, decision theory, imprecise probabilities, etc.
- A central notion related to TU-games is the *core*: it has its counterpart in all of the above mentioned fields.

Introduction

- Cooperative games with transferable utility (*TU-games*) are merely set functions on a finite set vanishing on the empty set.
- This wide definition makes them appear in various fields of discrete mathematics: combinatorial optimization and operations research, game theory, decision theory, imprecise probabilities, etc.
- A central notion related to TU-games is the *core*: it has its counterpart in all of the above mentioned fields.
- Games with a nonempty core are the *balanced games*, where the key notion behind is the notion of *balanced collection of sets*.

Introduction

- Cooperative games with transferable utility (*TU-games*) are merely set functions on a finite set vanishing on the empty set.
- This wide definition makes them appear in various fields of discrete mathematics: combinatorial optimization and operations research, game theory, decision theory, imprecise probabilities, etc.
- A central notion related to TU-games is the *core*: it has its counterpart in all of the above mentioned fields.
- Games with a nonempty core are the *balanced games*, where the key notion behind is the notion of *balanced collection of sets*.
- This talk is about balanced collections and balanced games, whose structure remains largely unexplored.

Outline

1. TU-games and the like
2. Balanced collections
3. Applications
4. Geometry of the set of balanced games

TU-games

- $N = \{1, \ldots, n\}$ set of players. Subsets of N are called *coalitions*.

TU-games

- $N = \{1, \ldots, n\}$ set of players. Subsets of N are called *coalitions*.
- A *game with transferable utility in characteristic form* (abbreviated by *TU-game* or simply *game*) is a mapping $v : 2^N \to \mathbb{R}$ s.t. $v(\varnothing) = 0$.

TU-games

- $N = \{1, \ldots, n\}$ set of players. Subsets of N are called *coalitions*.
- A *game with transferable utility in characteristic form* (abbreviated by *TU-game* or simply *game*) is a mapping $v : 2^N \to \mathbb{R}$ s.t. $v(\varnothing) = 0$.
- $x \in \mathbb{R}^N$ is a *payoff vector*. Notation: for every $S \subseteq N$,

$$x(S) = \sum_{i \in S} x_i$$

TU-games

- $N = \{1, \dots, n\}$ set of players. Subsets of N are called *coalitions*.
- A *game with transferable utility in characteristic form* (abbreviated by *TU-game* or simply *game*) is a mapping $v : 2^N \to \mathbb{R}$ s.t. $v(\varnothing) = 0$.
- $x \in \mathbb{R}^N$ is a *payoff vector*. Notation: for every $S \subseteq N$,

$$x(S) = \sum_{i \in S} x_i$$

- Aim of (cooperative) game theory: find a (set of) rational, satisfactory payoff vector(s) x, called the *solution* of the game. Usually, one impose $x(N) = v(N)$ (*efficiency: share the whole cake).*

TU-games

- $N = \{1, \ldots, n\}$ set of players. Subsets of N are called *coalitions*.
- A *game with transferable utility in characteristic form* (abbreviated by *TU-game* or simply *game*) is a mapping $v : 2^N \to \mathbb{R}$ s.t. $v(\varnothing) = 0$.
- $x \in \mathbb{R}^N$ is a *payoff vector*. Notation: for every $S \subseteq N$,

$$x(S) = \sum_{i \in S} x_i$$

- Aim of (cooperative) game theory: find a (set of) rational, satisfactory payoff vector(s) x, called the *solution* of the game. Usually, one impose $x(N) = v(N)$ (*efficiency: share the whole cake).*
- One of the best known solution: the *core* (Gillies, 1953)

$$C(v) = \{x \in \mathbb{R}^N : x(S) \geq v(S) \forall S, x(N) = v(N)\}$$

(coalitional rationality, or stability of the grand coalition N)

TU-games in other domains

- In decision theory, one considers *capacities*, which are monotone games: v is a capacity if $S \subseteq T$ implies $v(S) \leqslant v(T)$ and $v(N) = 1$.

TU-games in other domains

- In decision theory, one considers *capacities*, which are monotone games: v is a capacity if $S \subseteq T$ implies $v(S) \leqslant v(T)$ and $v(N) = 1$.
- The *lower envelope* of a convex set of probability measures is a capacity (*imprecise probabilities* (Walley, 1991)).

TU-games in other domains

- In decision theory, one considers *capacities*, which are monotone games: v is a capacity if $S \subseteq T$ implies $v(S) \leqslant v(T)$ and $v(N) = 1$.
- The *lower envelope* of a convex set of probability measures is a capacity (*imprecise probabilities* (Walley, 1991)).
- Probability measures are additive capacities: $v(A \cup B) = v(A) + v(B)$ for disjoint A, B

TU-games in other domains

- In decision theory, one considers *capacities*, which are monotone games: v is a capacity if $S \subseteq T$ implies $v(S) \leqslant v(T)$ and $v(N) = 1$.
- The *lower envelope* of a convex set of probability measures is a capacity (*imprecise probabilities* (Walley, 1991)).
- Probability measures are additive capacities: $v(A \cup B) = v(A) + v(B)$ for disjoint A, B
- The core of a capacity v is:

$$C(v) = \{x \in \mathbb{R}^N \,:\, x(S) \geq v(S) \forall S, x(N) = 1\}$$

i.e., $x \in C(v)$ can be interpreted as a probability measure dominating (compatible with) v.

TU-games in other domains

- In decision theory, one considers *capacities*, which are monotone games: v is a capacity if $S \subseteq T$ implies $v(S) \leqslant v(T)$ and $v(N) = 1$.
- The *lower envelope* of a convex set of probability measures is a capacity (*imprecise probabilities* (Walley, 1991)).
- Probability measures are additive capacities: $v(A \cup B) = v(A) + v(B)$ for disjoint A, B
- The core of a capacity v is:

$$C(v) = \{x \in \mathbb{R}^N \,:\, x(S) \geq v(S) \forall S, x(N) = 1\}$$

  i.e., $x \in C(v)$ can be interpreted as a probability measure dominating (compatible with) v.
- In combinatorial optimization, when v is submodular, it can be seen as the rank function of a matroid. Then the (anti-)core of v is the *base polyhedron of v* (Edmonds, 1970).

Outline

1. TU-games and the like
2. **Balanced collections**
3. Applications
4. Geometry of the set of balanced games

Balanced collections

- (Shapley, 1967) A collection $\mathcal{B} \subseteq 2^N$ of nonempty coalitions is called *balanced* if there exist positive numbers λ_S for all $S \in \mathcal{B}$ s.t.
$$\sum_{S \in \mathcal{B}} \lambda_S 1^S = 1^N$$
(i.e., for every $i \in N$, $\sum_{S \ni i, S \in \mathcal{B}} \lambda_S = 1$)(1^N is in the relative interior of the cone generated by the 1^S, $S \in \mathcal{B}$).

Balanced collections

- (Shapley, 1967) A collection $\mathcal{B} \subseteq 2^N$ of nonempty coalitions is called *balanced* if there exist positive numbers λ_S for all $S \in \mathcal{B}$ s.t.
$$\sum_{S \in \mathcal{B}} \lambda_S 1^S = 1^N$$
(i.e., for every $i \in N$, $\sum_{S \ni i, S \in \mathcal{B}} \lambda_S = 1$)($1^N$ is in the relative interior of the cone generated by the 1^S, $S \in \mathcal{B}$).
- $(\lambda_S)_{S \in \mathcal{B}}$ are the *balancing weights*.

Balanced collections

- (Shapley, 1967) A collection $\mathcal{B} \subseteq 2^N$ of nonempty coalitions is called *balanced* if there exist positive numbers λ_S for all $S \in \mathcal{B}$ s.t.
$$\sum_{S\in\mathcal{B}} \lambda_S 1^S = 1^N$$
(i.e., for every $i \in N$, $\sum_{S\ni i, S\in\mathcal{B}} \lambda_S = 1$)(1^N is in the relative interior of the cone generated by the 1^S, $S \in \mathcal{B}$).
- $(\lambda_S)_{S\in\mathcal{B}}$ are the *balancing weights*.
- Examples:

Balanced collections

- (Shapley, 1967) A collection $\mathcal{B} \subseteq 2^N$ of nonempty coalitions is called *balanced* if there exist positive numbers λ_S for all $S \in \mathcal{B}$ s.t.
$$\sum_{S\in\mathcal{B}} \lambda_S 1^S = 1^N$$
(i.e., for every $i \in N$, $\sum_{S\ni i, S\in\mathcal{B}} \lambda_S = 1$)(1^N is in the relative interior of the cone generated by the 1^S, $S \in \mathcal{B}$).
- $(\lambda_S)_{S\in\mathcal{B}}$ are the *balancing weights*.
- Examples:
  - Every partition (balancing weights: 1)

Balanced collections

- (Shapley, 1967) A collection $\mathcal{B} \subseteq 2^N$ of nonempty coalitions is called *balanced* if there exist positive numbers λ_S for all $S \in \mathcal{B}$ s.t.

$$\sum_{S \in \mathcal{B}} \lambda_S 1^S = 1^N$$

  (i.e., for every $i \in N$, $\sum_{S \ni i, S \in \mathcal{B}} \lambda_S = 1$)($1^N$ is in the relative interior of the cone generated by the 1^S, $S \in \mathcal{B}$).
- $(\lambda_S)_{S \in \mathcal{B}}$ are the *balancing weights*.
- Examples:
  - Every partition (balancing weights: 1)
  - $n = 3$: $\{\overline{12}, \overline{13}, \overline{23}\}$ with $\lambda = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

Balanced collections

- (Shapley, 1967) A collection $\mathcal{B} \subseteq 2^N$ of nonempty coalitions is called *balanced* if there exist positive numbers λ_S for all $S \in \mathcal{B}$ s.t.
$$\sum_{S \in \mathcal{B}} \lambda_S 1^S = 1^N$$
(i.e., for every $i \in N$, $\sum_{S \ni i, S \in \mathcal{B}} \lambda_S = 1$)($1^N$ is in the relative interior of the cone generated by the 1^S, $S \in \mathcal{B}$).
- $(\lambda_S)_{S \in \mathcal{B}}$ are the *balancing weights*.
- Examples:
  - Every partition (balancing weights: 1)
  - $n = 3$: $\{\overline{12}, \overline{13}, \overline{23}\}$ with $\lambda = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$
  - $n = 4$: $\{\overline{12}, \overline{13}, \overline{14}, \overline{234}\}$ with $\lambda = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right)$.

Balanced collections

- (Shapley, 1967) A collection $\mathcal{B} \subseteq 2^N$ of nonempty coalitions is called *balanced* if there exist positive numbers λ_S for all $S \in \mathcal{B}$ s.t.
$$\sum_{S \in \mathcal{B}} \lambda_S 1^S = 1^N$$
(i.e., for every $i \in N$, $\sum_{S \ni i, S \in \mathcal{B}} \lambda_S = 1$)($1^N$ is in the relative interior of the cone generated by the 1^S, $S \in \mathcal{B}$).
- $(\lambda_S)_{S \in \mathcal{B}}$ are the *balancing weights*.
- Examples:
  - Every partition (balancing weights: 1)
  - $n = 3$: $\{\overline{12}, \overline{13}, \overline{23}\}$ with $\lambda = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$
  - $n = 4$: $\{\overline{12}, \overline{13}, \overline{14}, \overline{234}\}$ with $\lambda = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right)$.
- A balanced collection is *minimal* if no proper subcollection is balanced (equivalently, the balancing weights are unique).

Balanced collections

- (Shapley, 1967) A collection $\mathcal{B} \subseteq 2^N$ of nonempty coalitions is called *balanced* if there exist positive numbers λ_S for all $S \in \mathcal{B}$ s.t.
$$\sum_{S \in \mathcal{B}} \lambda_S 1^S = 1^N$$
(i.e., for every $i \in N$, $\sum_{S \ni i, S \in \mathcal{B}} \lambda_S = 1$)($1^N$ is in the relative interior of the cone generated by the 1^S, $S \in \mathcal{B}$).
- $(\lambda_S)_{S \in \mathcal{B}}$ are the *balancing weights*.
- Examples:
  - Every partition (balancing weights: 1)
  - $n = 3$: $\{\overline{12}, \overline{13}, \overline{23}\}$ with $\lambda = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
  - $n = 4$: $\{\overline{12}, \overline{13}, \overline{14}, \overline{234}\}$ with $\lambda = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3})$.
- A balanced collection is *minimal* if no proper subcollection is balanced (equivalently, the balancing weights are unique).
- So far, the number of minimal balanced collections (m.b.c.) is unknown beyond $n = 4$. A recursive algorithm has been proposed by Peleg (1965).

What about unbalanced collections?

- A collection of subsets of N which does not contain a balanced collection is said to be *unbalanced*. It is *maximal* if no supercollection of it is unbalanced (Billera et al., 2012).

What about unbalanced collections?

- A collection of subsets of N which does not contain a balanced collection is said to be *unbalanced*. It is *maximal* if no supercollection of it is unbalanced (Billera et al., 2012).
- Equivalently, 1^N is *not* in the cone generated by the 1^S, $S \in \mathcal{B}$.

What about unbalanced collections?

- A collection of subsets of N which does not contain a balanced collection is said to be *unbalanced*. It is *maximal* if no supercollection of it is unbalanced (Billera et al., 2012).
- Equivalently, 1^N is *not* in the cone generated by the 1^S, $S \in \mathcal{B}$.
- unbalanced $\rightarrow$ not balanced, but not the converse!

What about unbalanced collections?

- A collection of subsets of N which does not contain a balanced collection is said to be *unbalanced*. It is *maximal* if no supercollection of it is unbalanced (Billera et al., 2012).
- Equivalently, 1^N is *not* in the cone generated by the 1^S, $S \in \mathcal{B}$.
- unbalanced $\rightarrow$ not balanced, but not the converse!
- What is known so far:

n	Nb of maximal unbalanced collections
2	2
3	6
4	32
5	370
6	11,292
7	1,066,044
8	347,326,352
9	419,172,756,930

- By Farkas Lemma, it can be shown that a collection $\mathcal{S}$ of nonempty sets is unbalanced if and only if there exists $y \in \mathbb{R}^N$ such that $\sum_{i \in N} y_i = 0$ and $\sum_{i \in S} y_i > 0$ for all $S \in \mathcal{S}$.

Unbalanced collections and hyperplanes arrangements

- By Farkas Lemma, it can be shown that a collection $\mathcal{S}$ of nonempty sets is unbalanced if and only if there exists $y \in \mathbb{R}^N$ such that $\sum_{i\in N} y_i = 0$ and $\sum_{i\in S} y_i > 0$ for all $S \in \mathcal{S}$.
- Examples:

Unbalanced collections and hyperplanes arrangements

- By Farkas Lemma, it can be shown that a collection $\mathcal{S}$ of nonempty sets is unbalanced if and only if there exists $y \in \mathbb{R}^N$ such that $\sum_{i\in N} y_i = 0$ and $\sum_{i\in S} y_i > 0$ for all $S \in \mathcal{S}$.
- Examples:
  (i) For $n = 3$: $\{\{1,2\},\{1,3\},\{1\}\}$, $y = (2,-1,-1)$;

Unbalanced collections and hyperplanes arrangements

- By Farkas Lemma, it can be shown that a collection $\mathcal{S}$ of nonempty sets is unbalanced if and only if there exists $y \in \mathbb{R}^N$ such that $\sum_{i\in N} y_i = 0$ and $\sum_{i\in S} y_i > 0$ for all $S \in \mathcal{S}$.
- Examples:
  (i) For $n = 3$: $\{\{1,2\},\{1,3\},\{1\}\}$, $y = (2,-1,-1)$;
  (ii) For $n = 4$: $\{\{1\},\{1,2\},\{1,3\},\{1,4\},\{1,2,3\},\{1,2,4\},\{1,3,4\}\}$, $y = (3,-1,-1,-1)$.

Unbalanced collections and hyperplanes arrangements

- By Farkas Lemma, it can be shown that a collection $\mathcal{S}$ of nonempty sets is unbalanced if and only if there exists $y \in \mathbb{R}^N$ such that $\sum_{i \in N} y_i = 0$ and $\sum_{i \in S} y_i > 0$ for all $S \in \mathcal{S}$.
- Examples:
  (i) For $n = 3$: $\{\{1,2\},\{1,3\},\{1\}\}$, $y = (2,-1,-1)$;
  (ii) For $n = 4$: $\{\{1\},\{1,2\},\{1,3\},\{1,4\},\{1,2,3\},\{1,2,4\},\{1,3,4\}\}$, $y = (3,-1,-1,-1)$.
- In the hyperplane $H_N = \{x \in \mathbb{R}^N \mid x(N) = 0\}$, consider the hyperplanes $\{x \in H_N \mid x(S) = 0\}$, for all $S \in 2^N \setminus \{\varnothing, N\}$ (only $2^{n-1} - 1$ distinct ones). There is a bijection between maximal unbalanced collections and regions induced by the hyperplane arrangement, which shows that maximal u.c. have $2^{n-1} - 1$ sets.

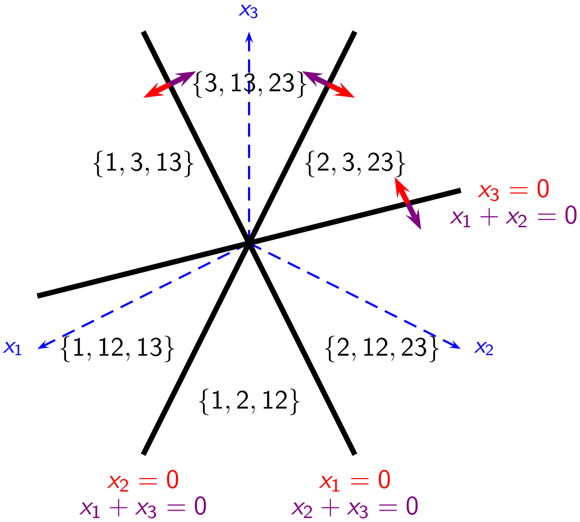


Figure: The restricted all-subset arrangement for $n = 3$ in the plane H_N. Arrows indicate the normal vector to the hyperplane of the same color. The 6 maximal unbalanced collections (subsets are written without comma and braces) correspond to the 6 regions.

Back to balanced collections: practical implementation

Laplace Mermoud, G. and Sudhölter (2023) implemented the Peleg algorithm in Python, and found the following:

Players	Minimal balanced collections	CPU time
1	1	-
2	2	$\sim$ 0.00 sec
3	6	$\sim$ 0.01 sec
4	42	$\sim$ 0.03 sec
5	1 292	$\sim$ 1.05 sec
6	200 214	$\sim$ 4 min 4 sec
7	132 422 036	$\sim$ 63 hours

Laplace Mermoud, G. and Sudhölter (2023) implemented the Peleg algorithm in Python, and found the following:

Players	Minimal balanced collections	CPU time
1	1	-
2	2	$\sim$ 0.00 sec
3	6	$\sim$ 0.01 sec
4	42	$\sim$ 0.03 sec
5	1 292	$\sim$ 1.05 sec
6	200 214	$\sim$ 4 min 4 sec
7	132 422 036	$\sim$ 63 hours

N.B. 1: This sequence is now in the OEIS (On Line Encyclopedia of Integer Sequences (Sloane, 1964)) under the number A355042 (see `https://oeis.org/A355042`).

Back to balanced collections: practical implementation

Laplace Mermoud, G. and Sudhölter (2023) implemented the Peleg algorithm in Python, and found the following:

Players	Minimal balanced collections	CPU time
1	1	-
2	2	$\sim$ 0.00 sec
3	6	$\sim$ 0.01 sec
4	42	$\sim$ 0.03 sec
5	1 292	$\sim$ 1.05 sec
6	200 214	$\sim$ 4 min 4 sec
7	132 422 036	$\sim$ 63 hours

N.B. 1: This sequence is now in the OEIS (On Line Encyclopedia of Integer Sequences (Sloane, 1964)) under the number A355042 (see `https://oeis.org/A355042`).

N.B. 2: We have stored the complete list of m.b.c. till $n = 7$

A comparison with a polyhedral approach

- Consider the polytope $W(N)$ defined by

$$W(N) = \left\{ \lambda \in \mathbb{R}^{2^N \setminus \{\emptyset\}} : \sum_{S \in 2^N \setminus \{\emptyset\}} \lambda_S 1^S = 1^N, \lambda_S \geq 0, \forall S \in 2^N \setminus \{\emptyset\} \right\}$$

A comparison with a polyhedral approach

- Consider the polytope $W(N)$ defined by

$$W(N) = \left\{ \lambda \in \mathbb{R}^{2^N \setminus \{\emptyset\}} : \sum_{S \in 2^N \setminus \{\emptyset\}} \lambda_S 1^S = 1^N, \lambda_S \geq 0, \forall S \in 2^N \setminus \{\emptyset\} \right\}$$

- It is easy to check that the vertices of $W(N)$ are in bijection with the minimal balanced collections on N, taking $\mathcal{B} := \{S : \lambda_S > 0\}$.

A comparison with a polyhedral approach

- Consider the polytope $W(N)$ defined by

$$W(N) = \left\{ \lambda \in \mathbb{R}^{2^N \setminus \{\emptyset\}} : \sum_{S \in 2^N \setminus \{\emptyset\}} \lambda_S 1^S = 1^N, \lambda_S \geq 0, \forall S \in 2^N \setminus \{\emptyset\} \right\}$$

- It is easy to check that the vertices of $W(N)$ are in bijection with the minimal balanced collections on N, taking $\mathcal{B} := \{S : \lambda_S > 0\}$.
- Consequently, generating all minimal balanced collections of N amounts to finding all vertices of $W(N)$.

A comparison with a polyhedral approach

- Consider the polytope $W(N)$ defined by

$$W(N) = \left\{ \lambda \in \mathbb{R}^{2^N \setminus \{\emptyset\}} : \sum_{S \in 2^N \setminus \{\emptyset\}} \lambda_S 1^S = 1^N, \lambda_S \geq 0, \forall S \in 2^N \setminus \{\emptyset\} \right\}$$

- It is easy to check that the vertices of $W(N)$ are in bijection with the minimal balanced collections on N, taking $\mathcal{B} := \{S : \lambda_S > 0\}$.
- Consequently, generating all minimal balanced collections of N amounts to finding all vertices of $W(N)$.
- This *vertex enumeration* problem can be solved by the Avis-Fukuda method (1992). Here are the CPU times when $n = 6$:

Peleg's algorithm	Avis-Fukuda algorithm
4mn 4s	29mn 24s

(pycddlib package used for Avis-Fukuda algorithm)

Outline

Nonemptiness of the core

Theorem (Bondareva-Shapley, sharp form)

A game v has a nonempty core if and only if for any minimal balanced collection $\mathcal{B}$ with balancing vector $(\lambda_S^{\mathcal{B}})_{S\in\mathcal{B}}$, we have

$$\sum_{S\in\mathcal{B}} \lambda_S^{\mathcal{B}} v(S) \leq v(N).$$

Moreover, none of the inequalities is redundant, except the one for $\mathcal{B} = \{N\}$.

Note: Games satisfying this condition are called *balanced*

Nonemptiness of the core

Theorem (Bondareva-Shapley, sharp form)

A game v has a nonempty core if and only if for any minimal balanced collection $\mathcal{B}$ with balancing vector $(\lambda_S^{\mathcal{B}})_{S\in\mathcal{B}}$, we have

$$\sum_{S\in\mathcal{B}} \lambda_S^{\mathcal{B}} v(S) \leq v(N).$$

Moreover, none of the inequalities is redundant, except the one for $\mathcal{B} = \{N\}$.

Note: Games satisfying this condition are called *balanced*

Equivalently, one can solve the following LP and check if the value of the LP is equal to $v(N)$:

$$\begin{aligned} \min \quad & x(N) \\ \text{s.t.} \quad & x(S) \geq v(S), \forall S \in 2^N \setminus \{\emptyset\} \end{aligned}$$

Comparison of CPU time (native simplex method available in Python), run on 5000 randomly chosen balanced TU-games with $n = 6$:

Bondareva-Shapley	LP
0.96s	24.85s

(Laplace Mermoud, G. and Sudhölter, 2023)

Comparison of CPU time (native simplex method available in Python), run on 5000 randomly chosen balanced TU-games with $n = 6$:

Bondareva-Shapley	LP
0.96s	24.85s

(Laplace Mermoud, G. and Sudhölter, 2023)

Important note: m.b.c. do not depend on the game, only on N. Hence they are generated only once for ever.

Outline

Effectiveness, exactness

- A coalition S is *exact* if $x(S) = v(S)$ for some $x \in C(v)$.

Effectiveness, exactness

- A coalition S is *exact* if $x(S) = v(S)$ for some $x \in C(v)$.
- A coalition S is *effective* if $x(S) = v(S)$ for all $x \in C(v)$. We denote by $\mathcal{E}(v)$ the set of effective coalitions.

Effectiveness, exactness

- A coalition S is *exact* if $x(S) = v(S)$ for some $x \in C(v)$.
- A coalition S is *effective* if $x(S) = v(S)$ for all $x \in C(v)$. We denote by $\mathcal{E}(v)$ the set of effective coalitions.
- For any coalition S we define the game (N, v^S) by

$$v^S(T) = \begin{cases} v(N) - v(S), & \text{if } T = N \setminus S \\ v(T), & \text{otherwise.} \end{cases}$$

Effectiveness, exactness

- A coalition S is *exact* if $x(S) = v(S)$ for some $x \in C(v)$.
- A coalition S is *effective* if $x(S) = v(S)$ for all $x \in C(v)$. We denote by $\mathcal{E}(v)$ the set of effective coalitions.
- For any coalition S we define the game (N, v^S) by

$$v^S(T) = \begin{cases} v(N) - v(S), & \text{if } T = N \setminus S \\ v(T), & \text{otherwise.} \end{cases}$$

Proposition (Laplace Mermoud, G. and Sudhölter, 2023)

Let v be a balanced game. Then a coalition S is exact iff (N, v^S) is balanced.

Effectiveness, exactness

- A coalition S is *exact* if $x(S) = v(S)$ for some $x \in C(v)$.
- A coalition S is *effective* if $x(S) = v(S)$ for all $x \in C(v)$. We denote by $\mathcal{E}(v)$ the set of effective coalitions.
- For any coalition S we define the game (N, v^S) by

$$v^S(T) = \begin{cases} v(N) - v(S), & \text{if } T = N \setminus S \\ v(T), & \text{otherwise.} \end{cases}$$

Proposition (Laplace Mermoud, G. and Sudhölter, 2023)

Let v be a balanced game. Then a coalition S is exact iff (N, v^S) is balanced.

Proposition (Laplace Mermoud, G. and Sudhölter, 2023)

$\mathcal{E}(v)$ is the union of all the minimal balanced collections $\mathcal{B}$ such that

$$\sum_{S \in \mathcal{B}} \lambda_S^{\mathcal{B}} v(S) = v(N).$$

Outline

Stable sets

- Let $X(v) = \{x \in \mathbb{R}^N \,:\, x(N) = v(N)\}$ and $I(v) = \{x \in X(v) \,:\, x_i \geqslant v(\{i\}), \forall i \in N\}$ (*imputations*)

Stable sets

- Let $X(v) = \{x \in \mathbb{R}^N : x(N) = v(N)\}$ and $I(v) = \{x \in X(v) : x_i \geqslant v(\{i\}), \forall i \in N\}$ (*imputations*)
- Let $x, y \in X(v)$ and $S \in 2^N \setminus \{\emptyset, N\}$. Then *x dominates y via S* ($x \operatorname{dom}_S y$) if:

Stable sets

- Let $X(v) = \{x \in \mathbb{R}^N : x(N) = v(N)\}$ and $I(v) = \{x \in X(v) : x_i \geqslant v(\{i\}), \forall i \in N\}$ (*imputations*)
- Let $x, y \in X(v)$ and $S \in 2^N \setminus \{\emptyset, N\}$. Then *x dominates y via S* ($x \mathrm{dom}_S y$) if:
  - $x_i > y_i, \forall i \in S$

- Let $X(v) = \{x \in \mathbb{R}^N : x(N) = v(N)\}$ and $I(v) = \{x \in X(v) : x_i \geqslant v(\{i\}), \forall i \in N\}$ (*imputations*)
- Let $x, y \in X(v)$ and $S \in 2^N \setminus \{\emptyset, N\}$. Then *$x$ dominates y via S* ($x \,\mathrm{dom}_S\, y$) if:
  - $x_i > y_i, \forall i \in S$
  - $x(S) \leq v(S)$

Stable sets

- Let $X(v) = \{x \in \mathbb{R}^N : x(N) = v(N)\}$ and $I(v) = \{x \in X(v) : x_i \geqslant v(\{i\}), \forall i \in N\}$ (*imputations*)
- Let $x, y \in X(v)$ and $S \in 2^N \setminus \{\emptyset, N\}$. Then *x dominates y via S* ($x \mathrm{dom}_S y$) if:
  - $x_i > y_i, \forall i \in S$
  - $x(S) \leq v(S)$
- *x dominates y* ($x \mathrm{dom} y$) if $x \mathrm{dom}_S y$ for some S

- Let $X(v) = \{x \in \mathbb{R}^N : x(N) = v(N)\}$ and $I(v) = \{x \in X(v) : x_i \geqslant v(\{i\}), \forall i \in N\}$ (*imputations*)
- Let $x, y \in X(v)$ and $S \in 2^N \setminus \{\emptyset, N\}$. Then *x dominates y via S* *($x \operatorname{dom}_S y$)* if:
  - $x_i > y_i, \forall i \in S$
  - $x(S) \leq v(S)$
- *x dominates y ($x \operatorname{dom} y$)* if $x \operatorname{dom}_S y$ for some S
- (Von Neumann and Morgenstern, 1944) $U \subseteq I(v)$ is a *stable set* if it satisfies

Stable sets

- Let $X(v) = \{x \in \mathbb{R}^N : x(N) = v(N)\}$ and $I(v) = \{x \in X(v) : x_i \geqslant v(\{i\}), \forall i \in N\}$ (*imputations*)
- Let $x, y \in X(v)$ and $S \in 2^N \setminus \{\emptyset, N\}$. Then *x dominates y via S* ($x \mathrm{dom}_S y$) if:
  - $x_i > y_i, \forall i \in S$
  - $x(S) \leq v(S)$
- *x dominates y* ($x \mathrm{dom}\ y$) if $x \mathrm{dom}_S y$ for some S
- (Von Neumann and Morgenstern, 1944) $U \subseteq I(v)$ is a *stable set* if it satisfies
  - *external stability:* $\forall y \notin U, \exists x \in U$ such that $x \mathrm{dom}\ y$;

Stable sets

- Let $X(v) = \{x \in \mathbb{R}^N : x(N) = v(N)\}$ and $I(v) = \{x \in X(v) : x_i \geqslant v(\{i\}), \forall i \in N\}$ (*imputations*)
- Let $x, y \in X(v)$ and $S \in 2^N \setminus \{\emptyset, N\}$. Then *x dominates y via S* ($x \mathrm{dom}_S\ y$) if:
  - $x_i > y_i, \forall i \in S$
  - $x(S) \leq v(S)$
- *x dominates y* ($x \mathrm{dom}\ y$) if $x \mathrm{dom}_S\ y$ for some S
- (Von Neumann and Morgenstern, 1944) $U \subseteq I(v)$ is a *stable set* if it satisfies
  - *external stability:* $\forall y \notin U, \exists x \in U$ such that $x \mathrm{dom}\ y$;
  - *internal stability:* $x \mathrm{dom}\ y\ \&\ y \in U \implies x \notin U$.

Stable sets

- Let $X(v) = \{x \in \mathbb{R}^N : x(N) = v(N)\}$ and $I(v) = \{x \in X(v) : x_i \geqslant v(\{i\}), \forall i \in N\}$ (*imputations*)
- Let $x, y \in X(v)$ and $S \in 2^N \setminus \{\emptyset, N\}$. Then *x dominates y via S* ($x \operatorname{dom}_S y$) if:
  - $x_i > y_i, \forall i \in S$
  - $x(S) \leq v(S)$
- *x dominates y* ($x \operatorname{dom} y$) if $x \operatorname{dom}_S y$ for some S
- (Von Neumann and Morgenstern, 1944) $U \subseteq I(v)$ is a *stable set* if it satisfies
  - *external stability:* $\forall y \notin U, \exists x \in U$ such that $x \operatorname{dom} y$;
  - *internal stability:* $x \operatorname{dom} y$ & $y \in U \implies x \notin U$.
- Stable sets may not exist, may be not unique, not convex...

Stable sets

- Let $X(v) = \{x \in \mathbb{R}^N : x(N) = v(N)\}$ and $I(v) = \{x \in X(v) : x_i \geqslant v(\{i\}), \forall i \in N\}$ (*imputations*)
- Let $x, y \in X(v)$ and $S \in 2^N \setminus \{\emptyset, N\}$. Then *x dominates y via S* ($x \operatorname{dom}_S y$) if:
  - $x_i > y_i, \forall i \in S$
  - $x(S) \leq v(S)$
- *x dominates y* ($x \operatorname{dom} y$) if $x \operatorname{dom}_S y$ for some S
- (Von Neumann and Morgenstern, 1944) $U \subseteq I(v)$ is a *stable set* if it satisfies
  - *external stability:* $\forall y \notin U, \exists x \in U$ such that $x \operatorname{dom} y$;
  - *internal stability:* $x \operatorname{dom} y \;\&\; y \in U \implies x \notin U$.
- Stable sets may not exist, may be not unique, not convex...
- If the core is stable, then it is the unique stable set of the game.

Stable sets

- Let $X(v) = \{x \in \mathbb{R}^N : x(N) = v(N)\}$ and $I(v) = \{x \in X(v) : x_i \geqslant v(\{i\}), \forall i \in N\}$ (*imputations*)
- Let $x, y \in X(v)$ and $S \in 2^N \setminus \{\emptyset, N\}$. Then *x dominates y via S* ($x \operatorname{dom}_S y$) if:
  - $x_i > y_i, \forall i \in S$
  - $x(S) \leq v(S)$
- *x dominates y* ($x \operatorname{dom} y$) if $x \operatorname{dom}_S y$ for some S
- (Von Neumann and Morgenstern, 1944) $U \subseteq I(v)$ is a *stable set* if it satisfies
  - *external stability:* $\forall y \notin U, \exists x \in U$ such that $x \operatorname{dom} y$;
  - *internal stability:* $x \operatorname{dom} y \;\&\; y \in U \implies x \notin U$.
- Stable sets may not exist, may be not unique, not convex...
- If the core is stable, then it is the unique stable set of the game.
- G. and Sudhölter (2021) found a (finite!) (but very combinatorial!!) necessary and sufficient condition for core stability using nested minimal balanced collections.

Why a nested balancedness condition?

- Assuming $C(v) \neq \emptyset$, the core is stable iff

$$\forall y \in X(v) \setminus C(v), \exists x(y) =: x \in C(v), \exists S \in 2^N, x_S \gg y_S, x(S) = v(S)$$

Why a nested balancedness condition?

- Assuming $C(v) \neq \emptyset$, the core is stable iff

$$\forall y \in X(v) \setminus C(v), \exists x(y) =: x \in C(v), \exists S \in 2^N, x_S \gg y_S, x(S) = v(S)$$

- 2 quantifiers on 2 variables in uncountable sets + linear inequalities

Why a nested balancedness condition?

- Assuming $C(v) \neq \emptyset$, the core is stable iff

$$\forall y \in X(v) \setminus C(v), \exists x(y) =: x \in C(v), \exists S \in 2^N, x_S \gg y_S, x(S) = v(S)$$

- 2 quantifiers on 2 variables in uncountable sets + linear inequalities
- Recall: test of nonemptiness of the core:

$$\exists x \in X(v), x(S) \geqslant v(S) \forall S$$

  (1 quantifier on 1 variable in an uncountable set + linear inequalities)

Why a nested balancedness condition?

- Assuming $C(v) \neq \emptyset$, the core is stable iff

$$\forall y \in X(v) \setminus C(v), \exists x(y) =: x \in C(v), \exists S \in 2^N, x_S \gg y_S, x(S) = v(S)$$

- 2 quantifiers on 2 variables in uncountable sets + linear inequalities
- Recall: test of nonemptiness of the core:

  $$\exists x \in X(v), x(S) \geqslant v(S) \forall S$$

  (1 quantifier on 1 variable in an uncountable set + linear inequalities)
- Bondareva-Shapley permits to reduce to a finite number of tests (=number of minimal balanced collections).

Why a nested balancedness condition?

- Assuming $C(v) \neq \emptyset$, the core is stable iff

$$\forall y \in X(v) \setminus C(v), \exists x(y) =: x \in C(v), \exists S \in 2^N, x_S \gg y_S, x(S) = v(S)$$

- 2 quantifiers on 2 variables in uncountable sets + linear inequalities
- Recall: test of nonemptiness of the core:

$$\exists x \in X(v), x(S) \geqslant v(S) \forall S$$

  (1 quantifier on 1 variable in an uncountable set + linear inequalities)
- Bondareva-Shapley permits to reduce to a finite number of tests (=number of minimal balanced collections).
- Taking a similar approach, we replace each test on a continuous variable by a finite balancedness condition

Why a nested balancedness condition?

- Assuming $C(v) \neq \emptyset$, the core is stable iff

$$\forall y \in X(v) \setminus C(v), \exists x(y) =: x \in C(v), \exists S \in 2^N, x_S \gg y_S, x(S) = v(S)$$

- 2 quantifiers on 2 variables in uncountable sets + linear inequalities
- Recall: test of nonemptiness of the core:

$$\exists x \in X(v), x(S) \geqslant v(S) \forall S$$

  (1 quantifier on 1 variable in an uncountable set + linear inequalities)
- Bondareva-Shapley permits to reduce to a finite number of tests (=number of minimal balanced collections).
- Taking a similar approach, we replace each test on a continuous variable by a finite balancedness condition

$\Rightarrow$ nested balancedness condition

Balanced sets

Definition

Let $Z \subseteq \mathbb{R}_+^N \setminus \{0\}$ be a finite set. $Z' \subseteq Z$ is a *balanced set* if there exists a nonnegative *balancing vector* $(\delta_z)_{z \in Z'}$ such that $\sum_{z \in Z'} \delta_z z = 1^N$.

Balanced sets

Definition

Let $Z \subseteq \mathbb{R}_+^N \setminus \{0\}$ be a finite set. $Z' \subseteq Z$ is a *balanced set* if there exists a nonnegative *balancing vector* $(\delta_z)_{z\in Z'}$ such that $\sum_{z\in Z'} \delta_z z = 1^N$.

- We say that a balanced subset is *minimal* if it does not contain a proper subset that is balanced (equivalently, if there is a unique balancing vector). Minimal balanced sets have at most n vectors.

Balanced sets

Definition

Let $Z \subseteq \mathbb{R}^N_+ \setminus \{0\}$ be a finite set. $Z' \subseteq Z$ is a *balanced set* if there exists a nonnegative *balancing vector* $(\delta_z)_{z \in Z'}$ such that $\sum_{z \in Z'} \delta_z z = 1^N$.

- We say that a balanced subset is *minimal* if it does not contain a proper subset that is balanced (equivalently, if there is a unique balancing vector). Minimal balanced sets have at most n vectors.
- Let $Z = \{z^1, \ldots, z^q\} \subseteq \mathbb{R}^N_+$ with $q \leqslant n$. Consider the matrix A^Z made by the column vectors $z^1, \ldots, z^q$. Then Z is a minimal balanced set if the following linear system in $\delta \in \mathbb{R}^q$

$$A^Z \delta = \mathbf{1}^N$$

has a unique solution which is positive.

Balanced sets

Definition

Let $Z \subseteq \mathbb{R}^N_+ \setminus \{0\}$ be a finite set. $Z' \subseteq Z$ is a *balanced set* if there exists a nonnegative *balancing vector* $(\delta_z)_{z \in Z'}$ such that $\sum_{z \in Z'} \delta_z z = 1^N$.

- We say that a balanced subset is *minimal* if it does not contain a proper subset that is balanced (equivalently, if there is a unique balancing vector). Minimal balanced sets have at most n vectors.
- Let $Z = \{z^1, \ldots, z^q\} \subseteq \mathbb{R}^N_+$ with $q \leqslant n$. Consider the matrix A^Z made by the column vectors $z^1, \ldots, z^q$. Then Z is a minimal balanced set if the following linear system in $\delta \in \mathbb{R}^q$

$$A^Z \delta = \mathbf{1}^N$$

has a unique solution which is positive.
- No specific algorithm for generating them so far...

Core stability: main result

Theorem (G. and Sudhölter, 2021)

Let (N, v) be a balanced game. Then v has a stable core if and only if for every feasible collection $\mathcal{S}$ and every $(\mathcal{B}_S)_{S\in\mathcal{S}} \in \mathbb{C}(\mathcal{S})$, either

$$\exists Z' \in \mathbb{B}(\mathcal{S}, (\mathcal{B}_S)_{S\in\mathcal{S}}) \setminus \mathbb{B}_0(\mathcal{S}, (\mathcal{B}_S)_{S\in\mathcal{S}}) : \sum_{z\in Z'} \delta_z^{Z'} a_z > v(N) \text{ holds or}$$

$$\exists Z' \in \mathbb{B}_0(\mathcal{S}, (\mathcal{B}_S)_{S\in\mathcal{S}}) : \sum_{z\in Z'} \delta_z^{Z'} a_z \geq v(N) \text{ holds.}$$

Outline

1. TU-games and the like
2. Balanced collections
3. Applications
4. **Geometry of the set of balanced games**

(joint work with P. Miranda and P. Garcia Segador)
Four sets are of interest:

Balanced games

(joint work with P. Miranda and P. Garcia Segador)
Four sets are of interest:

1. The set $\mathcal{BG}(n)$ of balanced games on $N = \{1, \ldots, n\}$

Balanced games

(joint work with P. Miranda and P. Garcia Segador)
Four sets are of interest:

1. The set $\mathcal{BG}(n)$ of balanced games on $N = \{1, \dots, n\}$
2. The set $\mathcal{BG}_\alpha(n)$ of balanced games v on N such that $v(N) = \alpha$

Balanced games

(joint work with P. Miranda and P. Garcia Segador)

Four sets are of interest:

1. The set $\mathcal{BG}(n)$ of balanced games on $N = \{1, \ldots, n\}$
2. The set $\mathcal{BG}_\alpha(n)$ of balanced games v on N such that $v(N) = \alpha$
3. The set $\mathcal{BG}_+(n)$ of balanced games v on N such that $v \geqslant 0$ (and $v(N) = 1$ arbitrarily fixed)

Balanced games

(joint work with P. Miranda and P. Garcia Segador)
Four sets are of interest:

1. The set $\mathcal{BG}(n)$ of balanced games on $N = \{1, \dots, n\}$
2. The set $\mathcal{BG}_\alpha(n)$ of balanced games v on N such that $v(N) = \alpha$
3. The set $\mathcal{BG}_+(n)$ of balanced games v on N such that $v \geqslant 0$ (and $v(N) = 1$ arbitrarily fixed)
4. The set $\mathcal{BG}_M(n)$ of balanced games which are monotone and $v(N) = 1$, i.e., capacities

Balanced games

(joint work with P. Miranda and P. Garcia Segador)
Four sets are of interest:

1. The set $\mathcal{BG}(n)$ of balanced games on $N = \{1, \ldots, n\}$
2. The set $\mathcal{BG}_\alpha(n)$ of balanced games v on N such that $v(N) = \alpha$
3. The set $\mathcal{BG}_+(n)$ of balanced games v on N such that $v \geqslant 0$ (and $v(N) = 1$ arbitrarily fixed)
4. The set $\mathcal{BG}_M(n)$ of balanced games which are monotone and $v(N) = 1$, i.e., capacities

The set $\mathcal{BG}_M(n)$ seems extremely difficult to study. Its structure is not elucidated.

Balanced games

(joint work with P. Miranda and P. Garcia Segador)
Four sets are of interest:

1. The set $\mathcal{BG}(n)$ of balanced games on $N = \{1, \ldots, n\}$
2. The set $\mathcal{BG}_\alpha(n)$ of balanced games v on N such that $v(N) = \alpha$
3. The set $\mathcal{BG}_+(n)$ of balanced games v on N such that $v \geqslant 0$ (and $v(N) = 1$ arbitrarily fixed)
4. The set $\mathcal{BG}_M(n)$ of balanced games which are monotone and $v(N) = 1$, i.e., capacities

The set $\mathcal{BG}_M(n)$ seems extremely difficult to study. Its structure is not elucidated.
$\rightarrow$ We focus on $\mathcal{BG}_+(n)$ and $\mathcal{BG}(n)$.

Balanced games

(joint work with P. Miranda and P. Garcia Segador)
Four sets are of interest:

1. The set $\mathcal{BG}(n)$ of balanced games on $N = \{1, \ldots, n\}$
2. The set $\mathcal{BG}_\alpha(n)$ of balanced games v on N such that $v(N) = \alpha$
3. The set $\mathcal{BG}_+(n)$ of balanced games v on N such that $v \geqslant 0$ (and $v(N) = 1$ arbitrarily fixed)
4. The set $\mathcal{BG}_M(n)$ of balanced games which are monotone and $v(N) = 1$, i.e., capacities

The set $\mathcal{BG}_M(n)$ seems extremely difficult to study. Its structure is not elucidated.
$\rightarrow$ We focus on $\mathcal{BG}_+(n)$ and $\mathcal{BG}(n)$.
Notation: $\mathfrak{B}^*(n)$: set of m.b.c. on N, except $\{N\}$.

Structure of $\mathcal{BG}_+(n)$

- $\mathcal{BG}_+(n)$ is determined by the following system of inequalities

$$\sum_{S \in \mathcal{B}} \lambda_S v(S) \leqslant 1, \quad \mathcal{B} \in \mathfrak{B}^*(n)$$
$$v(S) \geqslant 0, \quad S \in 2^N \setminus \{\varnothing, N\}$$

Structure of $\mathcal{BG}_+(n)$

- $\mathcal{BG}_+(n)$ is determined by the following system of inequalities

$$\sum_{S \in \mathcal{B}} \lambda_S v(S) \leqslant 1, \quad \mathcal{B} \in \mathfrak{B}^*(n)$$
$$v(S) \geqslant 0, \quad S \in 2^N \setminus \{\varnothing, N\}$$

- $\hookrightarrow$ $\mathcal{BG}_+(n)$ is a convex polytope. What are its vertices?

Structure of $\mathcal{BG}_+(n)$

- $\mathcal{BG}_+(n)$ is determined by the following system of inequalities
$$\sum_{S\in\mathcal{B}} \lambda_S v(S) \leqslant 1, \quad \mathcal{B} \in \mathfrak{B}^*(n)$$
$$v(S) \geqslant 0, \quad S \in 2^N \setminus \{\varnothing, N\}$$
- $\hookrightarrow$ $\mathcal{BG}_+(n)$ is a convex polytope. What are its vertices?

Theorem

v is a vertex of $\mathcal{BG}_+(n)$ if and only if v is balanced and 0-1-valued.

Structure of $\mathcal{BG}_+(n)$

- $\mathcal{BG}_+(n)$ is determined by the following system of inequalities

$$\sum_{S\in\mathcal{B}} \lambda_S v(S) \leqslant 1, \quad \mathcal{B} \in \mathfrak{B}^*(n)$$
$$v(S) \geqslant 0, \quad S \in 2^N \setminus \{\varnothing, N\}$$

- $\hookrightarrow \mathcal{BG}_+(n)$ is a convex polytope. What are its vertices?

Theorem

v is a vertex of $\mathcal{BG}_+(n)$ if and only if v is balanced and 0-1-valued.

Theorem

$v \in \mathcal{BG}_+(n)$ is a vertex iff either $v = 0$ or it has the following form:

$$v(S) = \begin{cases} 1, & \text{if } S \in \mathcal{D} \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathcal{D} \subseteq 2^N$ such that $\bigcap \mathcal{D} \neq \varnothing$.

Structure of $\mathcal{BG}_+(n)$

Lemma

Consider a vertex v of $\mathcal{BG}_+(n)$, *associated to collection* $\mathcal{D}$. *Then the dimension of the core of v is* $|\bigcap \mathcal{D}| - 1$

Structure of $\mathcal{BG}_+(n)$

Lemma

Consider a vertex v of $\mathcal{BG}_+(n)$, associated to collection $\mathcal{D}$. Then the dimension of the core of v is $|\bigcap\mathcal{D}| - 1$

Consequently, when $\bigcap\mathcal{D} = \{i\}$, the core is reduced to the vector $1^{\{i\}}$, i.e., the vector in $\mathbb{R}^n$ with ith component equal to 1, and 0 otherwise.

Structure of $\mathcal{BG}_+(n)$

Lemma

Consider a vertex v of $\mathcal{BG}_+(n)$, associated to collection $\mathcal{D}$. Then the dimension of the core of v is $|\bigcap \mathcal{D}| - 1$

Consequently, when $\bigcap \mathcal{D} = \{i\}$, the core is reduced to the vector $1^{\{i\}}$, i.e., the vector in $\mathbb{R}^n$ with ith component equal to 1, and 0 otherwise.

Theorem

The number of vertices v_n of $\mathcal{BG}_+(n)$ is given by $v_n = f_n + 1$ where f_n is defined recursively as follows:

$$f_n = \sum_{k=1}^{n-1} \binom{n}{k} \left(2^{2^k - 1} - f_k - 1\right), \forall n > 1 \text{ and } f_1 = 0$$

Structure of $\mathcal{BG}_+(n)$

Lemma

Consider a vertex v of $\mathcal{BG}_+(n)$*, associated to collection* $\mathcal{D}$*. Then the dimension of the core of v is* $|\bigcap \mathcal{D}| - 1$

Consequently, when $\bigcap \mathcal{D} = \{i\}$, the core is reduced to the vector $1^{\{i\}}$, i.e., the vector in $\mathbb{R}^n$ with ith component equal to 1, and 0 otherwise.

Theorem

The number of vertices v_n *of* $\mathcal{BG}_+(n)$ *is given by* $v_n = f_n + 1$ *where* f_n *is defined recursively as follows:*

$$f_n = \sum_{k=1}^{n-1} \binom{n}{k} \left(2^{2^k - 1} - f_k - 1\right), \forall n > 1 \text{ and } f_1 = 0$$

n	1	2	3	4	5	6	7	8
v_n	1	3	19	471	162631	12884412819	$6.456e+19$	$1.361e+39$

Structure of $\mathcal{BG}(n)$

- $\mathcal{BG}(n)$ is determined by the following system of inequalities
$$\sum_{S\in\mathcal{B}} \lambda_S v(S) - v(N) \leqslant 0, \quad \mathcal{B} \in \mathfrak{B}^*(n)$$

Structure of $\mathcal{BG}(n)$

- $\mathcal{BG}(n)$ is determined by the following system of inequalities
$$\sum_{S \in \mathcal{B}} \lambda_S v(S) - v(N) \leqslant 0, \quad \mathcal{B} \in \mathfrak{B}^*(n)$$
- $\hookrightarrow$ $\mathcal{BG}(n)$ is an unbounded convex polyhedron.

Structure of $\mathcal{BG}(n)$

- $\mathcal{BG}(n)$ is determined by the following system of inequalities
$$\sum_{S\in\mathcal{B}} \lambda_S v(S) - v(N) \leqslant 0, \quad \mathcal{B} \in \mathfrak{B}^*(n)$$
- $\hookrightarrow$ $\mathcal{BG}(n)$ is an unbounded convex polyhedron.
- For any nonempty $S \subseteq N$, we define

Structure of $\mathcal{BG}(n)$

- $\mathcal{BG}(n)$ is determined by the following system of inequalities
$$\sum_{S\in\mathcal{B}} \lambda_S v(S) - v(N) \leqslant 0, \quad \mathcal{B} \in \mathfrak{B}^*(n)$$
- $\hookrightarrow$ $\mathcal{BG}(n)$ is an unbounded convex polyhedron.
- For any nonempty $S \subseteq N$, we define
  - the *unanimity game* u_S by $u_S(T) = 1$ iff $T \supseteq S$ and 0 otherwise

Structure of $\mathcal{BG}(n)$

- $\mathcal{BG}(n)$ is determined by the following system of inequalities
$$\sum_{S\in\mathcal{B}} \lambda_S v(S) - v(N) \leqslant 0, \quad \mathcal{B} \in \mathfrak{B}^*(n)$$
- $\hookrightarrow$ $\mathcal{BG}(n)$ is an unbounded convex polyhedron.
- For any nonempty $S \subseteq N$, we define
  - the *unanimity game* u_S by $u_S(T) = 1$ iff $T \supseteq S$ and 0 otherwise
  - the *Dirac game* δ_S by $\delta_S(T) = 1$ iff $T = S$ and 0 otherwise

Structure of $\mathcal{BG}(n)$

- $\mathcal{BG}(n)$ is determined by the following system of inequalities
$$\sum_{S \in \mathcal{B}} \lambda_S v(S) - v(N) \leqslant 0, \quad \mathcal{B} \in \mathfrak{B}^*(n)$$
- $\hookrightarrow$ $\mathcal{BG}(n)$ is an unbounded convex polyhedron.
- For any nonempty $S \subseteq N$, we define
  - the *unanimity game* u_S by $u_S(T) = 1$ iff $T \supseteq S$ and 0 otherwise
  - the *Dirac game* δ_S by $\delta_S(T) = 1$ iff $T = S$ and 0 otherwise

Theorem

Let $n \geqslant 2$. Then $\mathcal{BG}(n)$ is $(2^n - 1)$-dimensional polyhedral cone, which is not pointed. Its lineality space $\mathrm{Lin}(\mathcal{BG}(n))$ has dimension n, with basis $(w_i)_{i \in N}$, $w_i = u_{\{i\}}$, the unanimity game centered on $\{i\}$

Structure of $\mathcal{BG}(n)$

- $\mathcal{BG}(n)$ is determined by the following system of inequalities
$$\sum_{S\in\mathcal{B}} \lambda_S v(S) - v(N) \leqslant 0, \quad \mathcal{B} \in \mathfrak{B}^*(n)$$
- $\hookrightarrow$ $\mathcal{BG}(n)$ is an unbounded convex polyhedron.
- For any nonempty $S \subseteq N$, we define
  - the *unanimity game* u_S by $u_S(T) = 1$ iff $T \supseteq S$ and 0 otherwise
  - the *Dirac game* δ_S by $\delta_S(T) = 1$ iff $T = S$ and 0 otherwise

Theorem

Let $n \geqslant 2$. Then $\mathcal{BG}(n)$ is $(2^n - 1)$-dimensional polyhedral cone, which is not pointed. Its lineality space $\mathrm{Lin}(\mathcal{BG}(n))$ has dimension n, with basis $(w_i)_{i\in N}$, $w_i = u_{\{i\}}$, the unanimity game centered on $\{i\}$

As $\mathcal{BG}(n)$ is not pointed, it can be decomposed as follows:

$$\mathcal{BG}(n) = \mathrm{Lin}(\mathcal{BG}(n)) \oplus \mathcal{BG}^0(n)$$

where $\mathcal{BG}^0(n)$ is a supplementary space (not unique), chosen so that the coordinates corresponding to singletons are zero.

Structure of $\mathcal{BG}(n)$

Theorem

Let $n \geqslant 2$. The extremal rays of $\mathcal{BG}(n)$ are

- *The $2n$ extremal rays corresponding to $\mathrm{Lin}(\mathcal{BG}(n))$: $w_1, \ldots, w_n, -w_1, \ldots, -w_n$;*
- *$2^n - n - 2$ extremal rays of the form $r_S = -\delta_S$, $S \subset N$, $|S| > 1$;*
- *n extremal rays of the form*

$$r_i = \sum_{S \ni i, |S|>1} \delta_S, \quad i \in N.$$

This yields in total $2^n + 2n - 2$ extremal rays.

Structure of $\mathcal{BG}(n)$

Theorem

Let $n \geqslant 2$. The extremal rays of $\mathcal{BG}(n)$ are

- *The $2n$ extremal rays corresponding to* $\mathrm{Lin}(\mathcal{BG}(n))$*:* $w_1, \ldots, w_n, -w_1, \ldots, -w_n$;
- $2^n - n - 2$ *extremal rays of the form* $r_S = -\delta_S$, $S \subset N$, $|S| > 1$;
- n *extremal rays of the form*

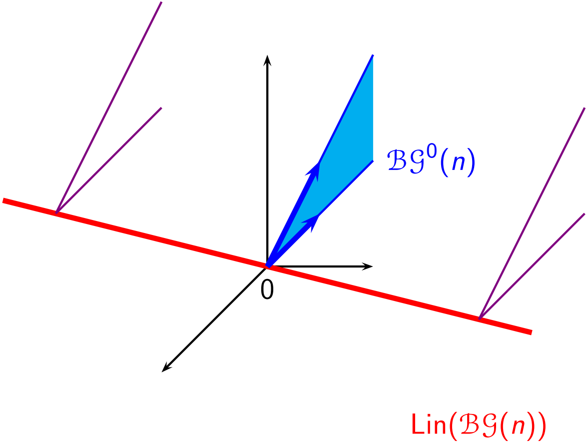
$$r_i = \sum_{S \ni i, |S|>1} \delta_S, \quad i \in N.$$

This yields in total $2^n + 2n - 2$ *extremal rays.*

Lemma

The cores of w_i, $-w_i$, r_i, r_S *for all* $i \in N$, $S \subset N$, $|S| > 1$ *are singletons (respectively,* $\{1^{\{i\}}\}$, $\{-1^{\{i\}}\}$, $\{1^{\{i\}}\}$, $\{0\}$*).*

Structure of $\mathcal{BG}(n)$



When is the core reduced to a point?

- In the case of $\mathcal{BG}(n)$, all extremal rays have a point core.

When is the core reduced to a point?

- In the case of $\mathcal{BG}(n)$, all extremal rays have a point core.
- However, in the case of $\mathcal{BG}_+(n)$, not all vertices have a point core: a vertex v has a point core iff its support $\mathcal{D}$ is s.t. $|\bigcap \mathcal{D}| = 1$.

When is the core reduced to a point?

General result: a game in the interior of $\mathcal{BG}_+(n)$ (or $\mathcal{BG}(n)$) does not have a point core.

When is the core reduced to a point?

General result: a game in the interior of $\mathcal{BG}_+(n)$ (or $\mathcal{BG}(n)$) does not have a point core.

Case of $\mathcal{BG}_+(n)$:

Lemma

Suppose v, v' are adjacent vertices of $\mathcal{BG}_+(n)$. Then a game on the edge defined by v, v' has a point core iff v, v' have a point core.

When is the core reduced to a point?

General result: a game in the interior of $\mathcal{BG}_+(n)$ (or $\mathcal{BG}(n)$) does not have a point core.

Case of $\mathcal{BG}_+(n)$:

Lemma

Suppose v, v' are adjacent vertices of $\mathcal{BG}_+(n)$. Then a game on the edge defined by v, v' has a point core iff v, v' have a point core.

More generally:

Lemma

Consider v in the relative interior of a p-dim face of $\mathcal{BG}_+(n)$. Then v has a point core iff all vertices defining the face have a point core.

When is the core reduced to a point? Case of $\mathcal{BG}(n)$

Lemma

Any game in the lineality space $\mathcal{BG}(n)$ has a point core.

When is the core reduced to a point? Case of $\mathcal{BG}(n)$

Lemma

Any game in the lineality space $\mathcal{BG}(n)$ has a point core.

We recall that facets of $\mathcal{BG}(n)$ are in bijection with the elements of $\mathfrak{B}^*(n)$, i.e., minimal balanced collections.

Theorem

Consider a m.b.c. $\mathcal{B} \in \mathfrak{B}^*(n)$ and its corresponding facet in $\mathcal{BG}(n)$.

1. *If $|\mathcal{B}| = n$, every game in the facet has a point core.*
2. *Otherwise, no game in the relative interior of the facet has a point core.*

When is the core reduced to a point? Case of $\mathcal{BG}(n)$

Lemma

Any game in the lineality space $\mathcal{BG}(n)$ has a point core.

We recall that facets of $\mathcal{BG}(n)$ are in bijection with the elements of $\mathfrak{B}^*(n)$, i.e., minimal balanced collections.

Theorem

Consider a m.b.c. $\mathcal{B} \in \mathfrak{B}^*(n)$ and its corresponding facet in $\mathcal{BG}(n)$.

1. *If $|\mathcal{B}| = n$, every game in the facet has a point core.*
2. *Otherwise, no game in the relative interior of the facet has a point core.*

Theorem

Consider a face $\mathcal{F}$ of $\mathcal{BG}(n)$, being the interection of facets $\mathcal{F}_1, \dots, \mathcal{F}_p$ with associated m.b.c. $\mathcal{B}_1, \dots, \mathcal{B}_p$. Then any game in $\mathcal{F}$ has a point core iff the rank of the matrix $\{1^S, S \in \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_p\}$ is n.

The case $n = 3$

The lineality space has basis $\{u_{\{1\}}, u_{\{2\}}, u_{\{3\}}\}$, with extremal rays $-\delta_{12}, -\delta_{13}, -\delta_{23}$, and r_1, r_2, r_3.

The case $n = 3$

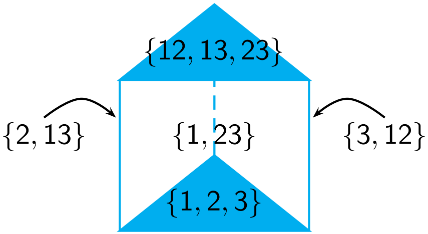
The lineality space has basis $\{u_{\{1\}}, u_{\{2\}}, u_{\{3\}}\}$, with extremal rays $-\delta_{12}, -\delta_{13}, -\delta_{23}$, and r_1, r_2, r_3.

m.b.c.	$-\delta_{12}$	$-\delta_{13}$	$-\delta_{23}$	r_1	r_2	r_3
$\mathcal{B}_1 = \{1, 2, 3\}$	×	×	×			
$\mathcal{B}_2 = \{1, 23\}$	×	×			×	×
$\mathcal{B}_3 = \{2, 13\}$	×		×	×		×
$\mathcal{B}_4 = \{3, 12\}$		×	×	×	×	
$\mathcal{B}_5 = \{12, 13, 23\}$				×	×	×

The case $n = 3$

The lineality space has basis $\{u_{\{1\}}, u_{\{2\}}, u_{\{3\}}\}$, with extremal rays $-\delta_{12}, -\delta_{13}, -\delta_{23}$, and r_1, r_2, r_3.

m.b.c.	$-\delta_{12}$	$-\delta_{13}$	$-\delta_{23}$	r_1	r_2	r_3
$\mathcal{B}_1 = \{1, 2, 3\}$	$\times$	$\times$	$\times$			
$\mathcal{B}_2 = \{1, 23\}$	$\times$	$\times$			$\times$	$\times$
$\mathcal{B}_3 = \{2, 13\}$	$\times$		$\times$	$\times$		$\times$
$\mathcal{B}_4 = \{3, 12\}$		$\times$	$\times$	$\times$	$\times$	
$\mathcal{B}_5 = \{12, 13, 23\}$				$\times$	$\times$	$\times$



That's all for the moment...

Thank you for your attention !