On the Relationship between Graphical and Compositional Models for the Dempster-Shafer Theory of Belief Functions

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Dempster's combination

using commonality functions for distinct BPAs:

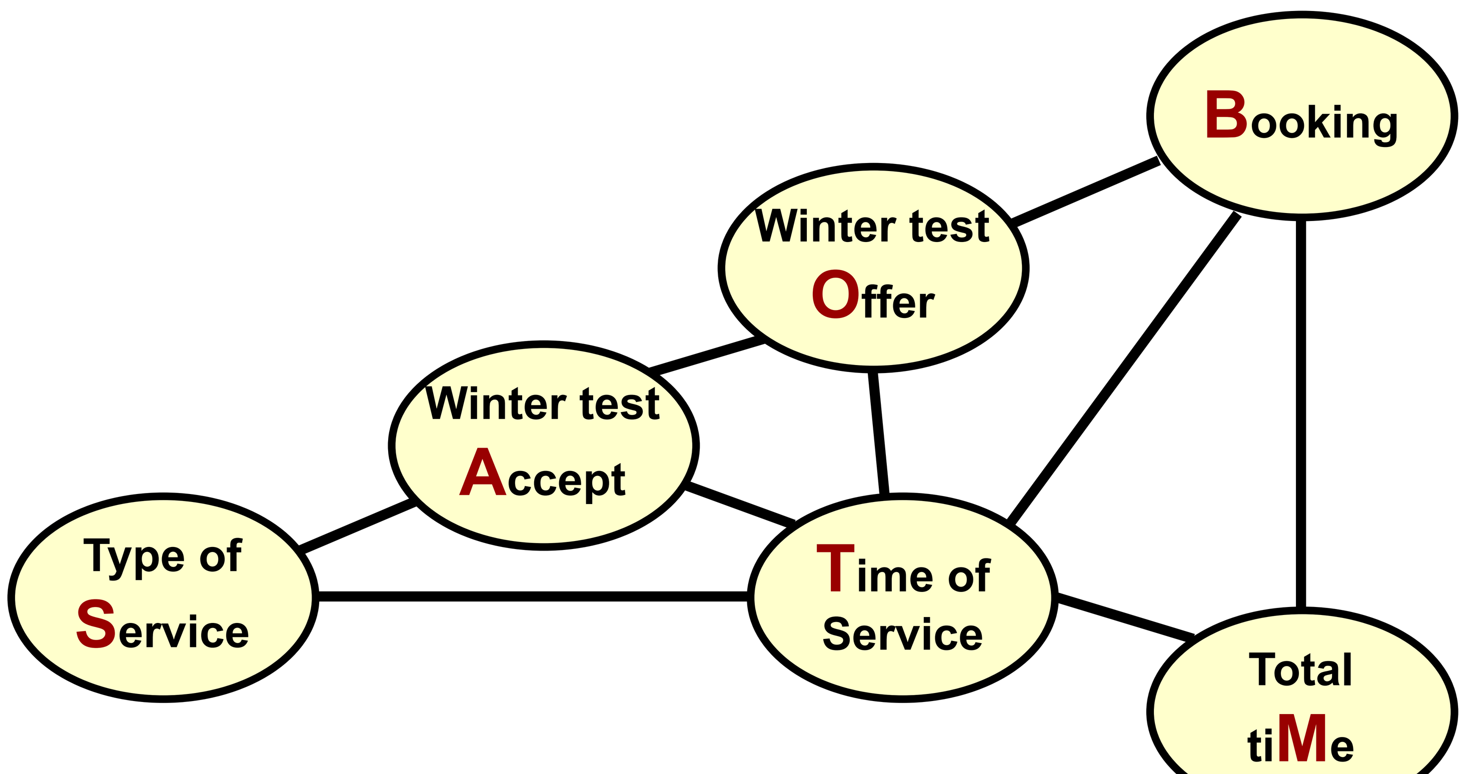
$$Q_{m_1 \oplus m_2}(\mathrm{a}) = \frac{1}{K}\, Q_{m_1}(\mathrm{a}^{\downarrow r})\, Q_{m_2}(\mathrm{a}^{\downarrow s})$$

Directed graphical model

DAG + system of conditional BPAs: $m_{X|Pa(X)}$

$$m = m_B \oplus m_S \oplus m_{O|B} \oplus m_{A|O} \oplus m_{T|A,S} \oplus m_{M|B,T}$$

Distinctness is guaranteed by combining "conditionals.":
For node X, $m^{\downarrow Pa(X)}_{\{X\}\cup pa(X)}$ is vacuous.



f-composition

Consider BPAs m_1 for r and m_2 for s. Their f-composition is a BPA $m_1 \triangleright_f m_2$ defined for each nonempty $\mathrm{a} \subseteq \Omega_{r\cup s}$ by one of the following expressions:

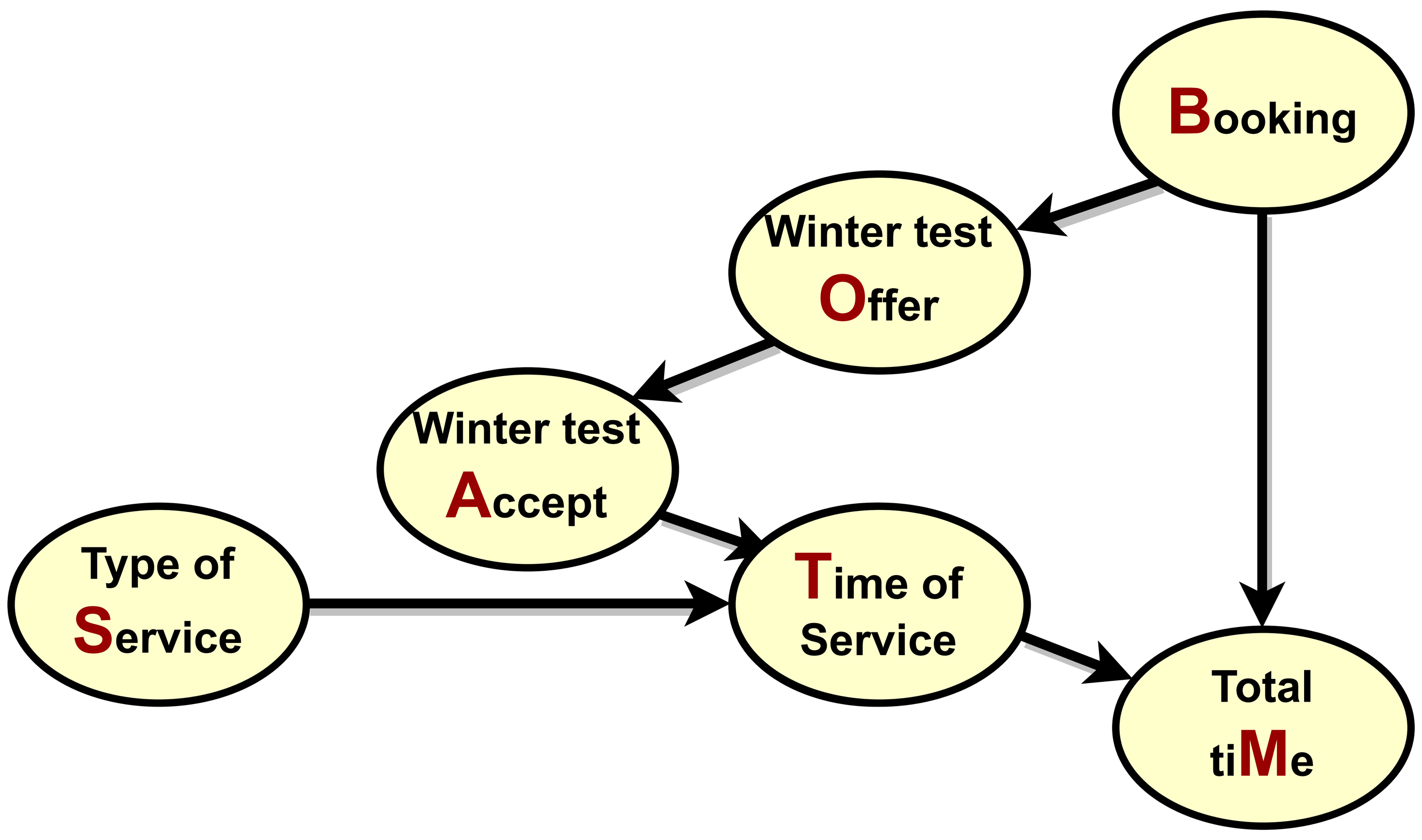
(i) if $m_2^{\downarrow r\cap s}(\mathrm{a}^{\downarrow r\cap s}) > 0$ and $\mathrm{a} = \mathrm{a}^{\downarrow r} \bowtie \mathrm{a}^{\downarrow s}$, then

$$(m_1 \triangleright_f m_2)(\mathrm{a}) = \frac{m_1(\mathrm{a}^{\downarrow r}) \cdot m_2(\mathrm{a}^{\downarrow s})}{m_2^{\downarrow r\cap s}(\mathrm{a}^{\downarrow r\cap s})};$$

(ii) if $m_2^{\downarrow r\cap s}(\mathrm{a}^{\downarrow r\cap s}) = 0$ and $\mathrm{a} = \mathrm{a}^{\downarrow r} \times \Omega_{s\setminus r}$, then $(m_1 \triangleright_f m_2)(\mathrm{a}) = m_1(\mathrm{a}^{\downarrow r})$;

(iii) in all other cases, $(m_1 \triangleright_f m_2)(\mathrm{a}) = 0$.

- f-composition is always defined
- There is a necessary and sufficient condition for $m_1 \triangleright_d m_2 = m_1 \triangleright_f m_2$ if the respective conditional $m_2 \ominus m_2^{\downarrow r\cap s}$ is non-negative.
- If d-composition is defined, then it is more specific than f-composition: $Bel(m_1 \triangleright_d m_2) \geq Bel(m_1 \triangleright_f m_2)$.
- f-composition can approximate the d-composition when it is undefined.



Dempster's decombination

The inverse of Dempster's combination, using commonality functions:

$$(Q_{m \ominus m^{\downarrow s}})(\mathrm{a}) = \begin{cases} \frac{1}{L} \frac{Q_m(\mathrm{a})}{Q_{m^{\downarrow s}}(\mathrm{a}^{\downarrow s})} & \text{if } Q_{m^{\downarrow s}}(\mathrm{a}) > 0 \\ !!! & \text{otherwise} \end{cases}$$

d-composition

$$m_1 \triangleright_d m_2 = m_1 \oplus (m_2 \ominus m_2^{\downarrow r\cap s})$$

Compositional models

Undirected graph G + system of BPAs: $m_{\mathcal{C}}$ for all cliques $\mathcal{C}$ of G connected using $\triangleright_d$ or $\triangleright_f$

$$\begin{aligned} m &= m_B \oplus m_S \oplus m_{O|B} \oplus m_{A|O} \oplus m_{T|A,S} \oplus m_{M|B,T} \\ &= m_B \triangleright_d m_S \triangleright_d m_{O|B} \triangleright_d m_{A|O} \triangleright_d m_{T|A,S} \triangleright_d m_{M|B,T} \\ &= m^{\downarrow B} \triangleright_d m^{\downarrow S} \triangleright_d m^{\downarrow O,B} \triangleright_d m^{\downarrow A,O} \triangleright_d m^{\downarrow T,A,S} \triangleright_d m^{\downarrow M,B,T} \end{aligned}$$

Open problems

- Computational problems of $\triangleright_d$. Is there a way to compute $m \ominus m^{\downarrow s}$ for some (non-trivial) class of BPAs avoiding the transformation of m into CF?
- Dempster's decombination is not unique. E.g. for BPA m, for which $\oplus$ is idempotent ($m \oplus m = m$) also $m \oplus \iota_m = m$.
- $m \ominus m^{\downarrow s}$ is often pseudo-BPA:
  - How to recognize when $m_1 \triangleright_d m_2$ is a non-negative BPA?
  - Do there exist some necessary and sufficient condition for $m_1 \triangleright_d m_2 = m_1 \triangleright_f m_2$ even when the respective conditional $m_2 \ominus m_2^{\downarrow r\cap s}$ is not non-negative.
- How to create conditional BPAs? Is there a possibility to computing conditional when $m \ominus m^{\downarrow s}$ is pseudo-BPAs