The logic FP(t,t) and two sorted equational states

S. Lapenta, S. Napolitano, L. Spada
Dep. of Mathematics - University of Salerno (Italy)

1. Introduction

Many-valued logic + modal logic = logic of probability

"$\Box\varphi$" reads as "φ is probable"
many valued formula — its truth value is the probability of φ

φ can be

- a boolean formula "the coin toss gives tail" FP(t)
- a many-valued formula "it will rain a lot" FP(t,t)

The logic FP(t) was introduced by Hájek, Godo and Esteva

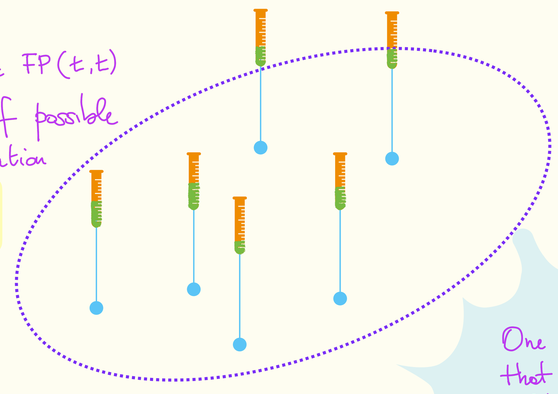
The logic FP(t,t) was introduced by Flaminio and Godo

We work in a fixed many-valued logic: Łukasiewicz logic and its algebraic semantics: MV-algebras.

2. Possible worlds semantics

A natural semantics for the logic FP(t,t) is given by a set of possible worlds W and an evaluation

$v : W \times Var \to [0,1]$



One can immagine that to each world it corresponds the outcome of an experiment.

The interpretation of the modality $\Box$ is given by a state.

States are the algebraic counterparts of expected values.

Flaminio proved that FP(t,t) is complete with regard to the above models

[T. Flaminio. On standard completeness and finite model property for a probabilistic logic on Łukasiewicz events. Journal of approximate reasoning 2021]

3. (Two sorted) algebraic semantics

Many sorted algebras are structures in which the domain is partitioned in sorts and operations may act within or across sorts.

E.g. Graphs can be seen as 2-sorted algebras in which the domain is partitioned in edges and nodes and there are two operations:

dom : edges $\to$ nodes cod : edges $\to$ nodes

Equational states [Kroupa-Mazzu] are two-sorted algebras (A_1, A_2) in which:

1) Each A_i is an MV-algebra (for $i = 1,2$)
2) there is an operation $s : A_1 \to A_2$ satisfying
   a) $s(1) = 1$
   b) $s(\neg a) = \neg s(a)$
   c) $s(a \oplus b) = s(a) \oplus s(b \wedge \neg a)$

Notice that equational states form a variety of algebras

4. Completeness

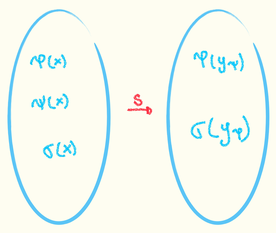
Main result

The logic FP(t,t) is complete with respect to equational states.

The proof proceeds in steps

1. Introduce new variables to encode the modal formulas into pure many-valued formulas.
2. Build a Lindenbaum-Tarski (two-sorted) algebra of formulas in the pure many valued logic.
3. Endow the algebra with an equational state
4. Show that if $FP(t,t) \not\vdash \varphi$ then the algebra just built makes the associated equation $\varphi \approx$ fail.

$\Box\varphi \rightsquigarrow y_\varphi$
$\Box\psi \rightsquigarrow y_\psi$



5. Free (equational) states

Corollary:

The algebra built in proof of completeness is in fact the free ω-generated equational state

The free generated equational states should be thought of as the most general probability assignment that can be given to an event

coin toss

$1/2 - 1/2$ Vs $LT_{eq}/\langle x^2 = x \rangle$

6. Conclusions and future works

1. Equational states provide an algebraic framework to study probability
2. This enable a wealth of algebraic tools to be used in probability
3. Our contribution establishes a deep link between equational states and the logic FP(t,t).

Much remains to be done:

Study the variety of equational states: subdirectly irreducible, generic algebra, functional representation, etc...

Use classical algebraic constructions to address paradoxes and open problems in probability.