Vertical Barrier Models as Unified Distortions

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Summary

Vertical Barrier Models are a class of distortion models that include some of the most important families in the literature, while still being relatively simple. In this paper we explore, in a finite framework, some facets of these models not yet investigated: their interpretation as neighbourhood models, the structure of their credal set in terms of maximum number of its extreme points, the result of merging operations with VBMs, and conditions for VBMs to be belief functions or possibility measures.

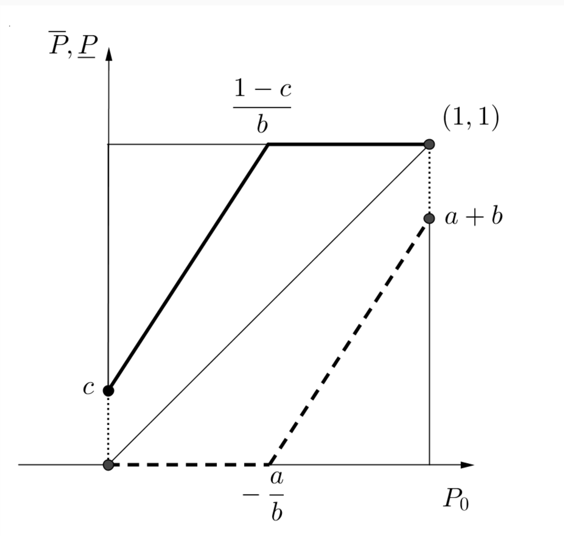
1. Introduction

Consider a probability measure $P_0 \in \mathbb{P}(\Omega)$ and two parameters a, b with $a \leq 0, b > 0$ and $a + b \in [0, 1]$. The associated Vertical Barrier Model (VBM) is identified by (P_0, a, b) and its lower probability $\underline{P}$ is given by

$$\underline{P}(A) = \max\{bP_0(A) + a, 0\} \ \forall A \subset \Omega \text{ and } \underline{P}(\Omega) = 1.$$

Its conjugate upper probability is

$$\overline{P}(A) = \min\{bP_0(A) + c, 1\} \ \forall A \neq \emptyset, \overline{P}(\emptyset) = 0, \text{ where } c = 1 - (a + b).$$



We denote by $\mathcal{M}(\underline{P}) := \{P \text{ prob. measure} \mid P(A) \geq \underline{P}(A) \ \forall A\}$ the associated credal set.

2. Distorting function of a VBM

Consider $a < 0 < b$ such that $a + b < 1$, and let us define $d_{VBM_{(a,b)}}$ on $\mathbb{P}(\Omega) \times \mathbb{P}(\Omega)$ by

$$d_{VBM}(P, P_0) = \max_{A \subseteq \Omega} \frac{P_0(A) - P(A)}{(1-b)P_0(A) - a}.$$

- d_{VBM} is well-defined and takes values in $[0, +\infty)$.
- $d_{VBM}(P, P_0) = 0 \Leftrightarrow P = P_0$.
- For every $\alpha \in [0, 1]$, $d_{VBM}(\alpha P_1 + (1-\alpha)P_2, P_0) \leq \max\{d_{VBM}(P_1, P_0), d_{VBM}(P_2, P_0)\}$.
- For every $P_0, P_1, P_2 \in \mathbb{P}(\Omega)$ and every $\epsilon > 0$ there is some $\delta > 0$ such that
$$||P_1 - P_2|| < \delta \Rightarrow |d_{VBM}(P_1, P_0) - d_{VBM}(P_2, P_0)| < \epsilon.$$

The VBM (P_0, a, b) can be seen as the neighbourhood model determined by d_{VBM} around P_0, in that:

$$\mathcal{M}(\underline{P}) = B^1_{d_{VBM}}(P_0) = \{P \in \mathbb{P}(\Omega) \mid d_{VBM}(P, P_0) \leq 1\}.$$

The definition of d_{VBM} can be extended to the case of $a = 0$ or $a + b = 1$ and then it gives the distorting functions of the PMM or the LV models.

3. Connection with other distortion models

- If $a < 0$ and $a + b = 1$ we obtain the pari-mutuel model (PMM).
- If $a = 0$ and $b < 1$ we obtain the linear-vacuous model (LV).
- If $a \in (-1, 0)$ and $b = 1$ we obtain the total variation model (TV).
- $\underline{P}$ is a VBM $\Leftrightarrow$ it is a convex combination of a PMM and a vacuous model.
- The L_1, Kolmogorov and constant-odds ratio models are **NOT** particular cases of VBM.

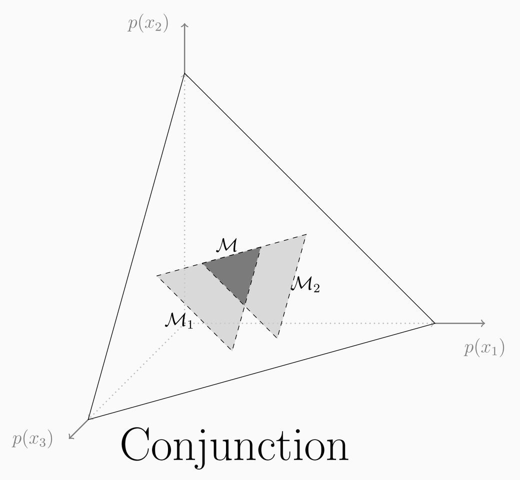
4. Structure of the credal set

- Given a VBM $\underline{P}$ on a space Ω of cardinality n, the maximum number of extreme points of $\mathcal{M}(\underline{P})$ is
$$\frac{n!}{\lfloor \frac{n}{2} - 1 \rfloor ! \lceil \frac{n}{2} - 1 \rceil !}.$$
- When in particular $\underline{P}(\{\omega\}) > 0, \forall \omega \in \Omega$, the maximum number of extreme points of $\mathcal{M}(\underline{P})$ is $n(n-1)$.
- The bounds are the same as for TV models, but greater than for PMM or LV models.

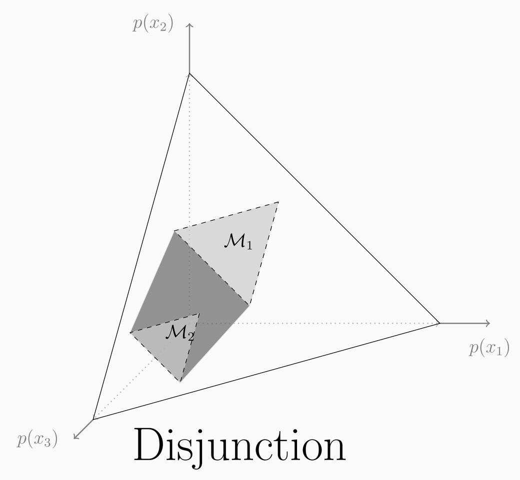
5. Processing Vertical Barrier Models

Given two VBMs $\underline{P}_1, \underline{P}_2$, we consider three different processing operations:

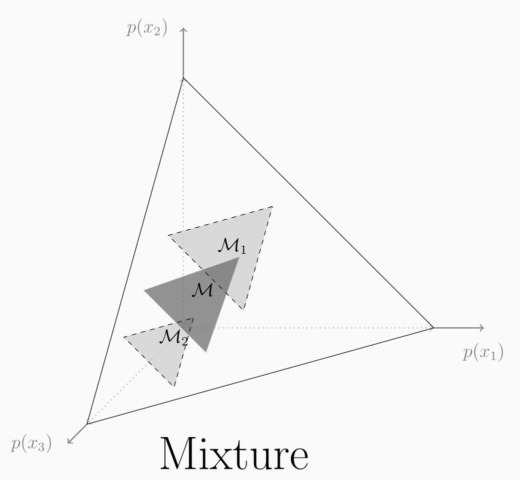
- Conjunction: $\underline{P}^{\cap}$ is the lower envelope of $\mathcal{M}(\underline{P}_1) \cap \mathcal{M}(\underline{P}_2)$.
- Disjunction: $\underline{P}^{\cup}$ is the lower envelope of $\mathcal{M}(\underline{P}_1) \cup \mathcal{M}(\underline{P}_2)$.
- Mixture: we fix $\alpha \in (0, 1)$ and take the lower envelope $\underline{P}^{\alpha}$ of $\alpha\mathcal{M}(\underline{P}_1) + (1-\alpha)\mathcal{M}(\underline{P}_2)$.



Conjunction



Disjunction



Mixture

- The conjunction or the disjunction of two VBMs is **NOT** a VBM in general.
- If $\{A : \underline{P}_1(A) = 0\} = \{A : \underline{P}_2(A) = 0\}$, then the mixture $\underline{P}^{\alpha}$ is also a VBM.
- However, a mixture of two VBMs is not a VBM in general.

6. VBMs, belief functions and possibility measures

- If $|\Omega| = 3$ and $b \leq 1$ then $\underline{P}$ is a belief function.
- If $\underline{P}$ is a belief function, then there cannot exist $A_1, A_2, A_3 \in \mathcal{P}(\Omega)$ such that $A_i \cap A_j = \emptyset$ $\forall i \neq j$, $\underline{P}(A_i) > 0 \ \forall i$ and $A_1 \cup A_2 \cup A_3 \neq \Omega$.

Thus, if $|\Omega|$ is large enough, a VBM will have to assign lower probability zero to the majority of events, and hence to be almost vacuous, for $\underline{P}$ to be a belief function.

Assume $P_0(\{\omega\}) > 0$ for every ω and let $k = \min\{|A| : \underline{P}(A) > 0\}$:

- The following are **sufficient** conditions for $\underline{P}$ to be a belief function:
  (i) $k = n - 1$ and $\sum_{\omega \in \Omega} \underline{P}(\{\omega\}^c) \leq 1$.
  (ii) $\exists! B$ such that $|B| = k < n - 1$ and $\underline{P}(A) > 0 \Leftrightarrow A \supseteq B$.
  (iii) $\exists! B$ such that $|B| = k - 1 < n - 2$, $bP_0(B) + a = 0$ and $\underline{P}(A) > 0 \Leftrightarrow A \supset B$.
- If $m(\Omega) \geq 1 - (a + b)$, then it is **necessary** that one of (i)–(iii) holds for $\underline{P}$ to be a belief function.
- On the other hand, $\overline{P}$ maxitive $\Leftrightarrow |\{\omega : \overline{P}(\{\omega\}) < 1\}| \leq 1$.

7. Comparison

	Conjunction	Disjunction	Mixture	$\underline{P}$ is a belief function?	$\max \lvert \text{ext}(\mathcal{M}(\underline{P})) \rvert$
PMM	YES	NO	Sometimes*	Sometimes**	$\frac{n!}{\lfloor\frac{n}{2}\rfloor \lfloor\frac{n}{2}-1\rfloor! \lceil\frac{n}{2}+1\rceil!}$
LV	YES	NO	YES	YES	n
TV	NO	NO	YES	NO	$\frac{n!}{\lfloor\frac{n}{2}-1\rfloor! \lceil\frac{n}{2}-1\rceil!}$
VBM	NO	NO	Sometimes*	Sometimes***	$\frac{n!}{\lfloor\frac{n}{2}-1\rfloor! \lceil\frac{n}{2}-1\rceil!}$

*In particular, if no zero lower probabilities are involved.

**There are necessary and sufficient conditions for $\underline{P}$ to be a belief function.

***There are sufficient conditions and also necessary ones.

Although they do not reduce to preexisting models in general, VMBs are not much more complex, as their distorting function and complexity of the credal set show.

Main references

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