A Study of Jeffrey’s Rule With Imprecise Probability Models

1 The setting

Given You have a finite possibility space Ω and a probability measure P on Ω.

New information You observe a new probability measure $\check{P}$ on a partition $\mathscr{B}$ of Ω.

Question How should you update your probability measure P taking into account this information? We are looking for a probability measure $\widehat{P}$ on Ω that satisfies the constraints

- $\widehat{P}(B) = \check{P}(B)$ for all B in $\mathscr{B}$, [agreeing on $\mathscr{B}$]
- $\widehat{P}(A|B) = P(A|B)$ for all B in $\mathscr{B}$ and $A \subseteq \Omega$. [rigidity]

Jeffrey’s Rule The unique probability measure $\widehat{P}$ on Ω that satisfies ‘agreeing on $\mathscr{B}$’ and ‘rigidity’ is given by

$$\widehat{P}(A) = \sum_{B \in \mathscr{B}} P(A|B)\check{P}(B) \quad \text{for all } A \subseteq \Omega.$$

2 Sets of desirable gambles

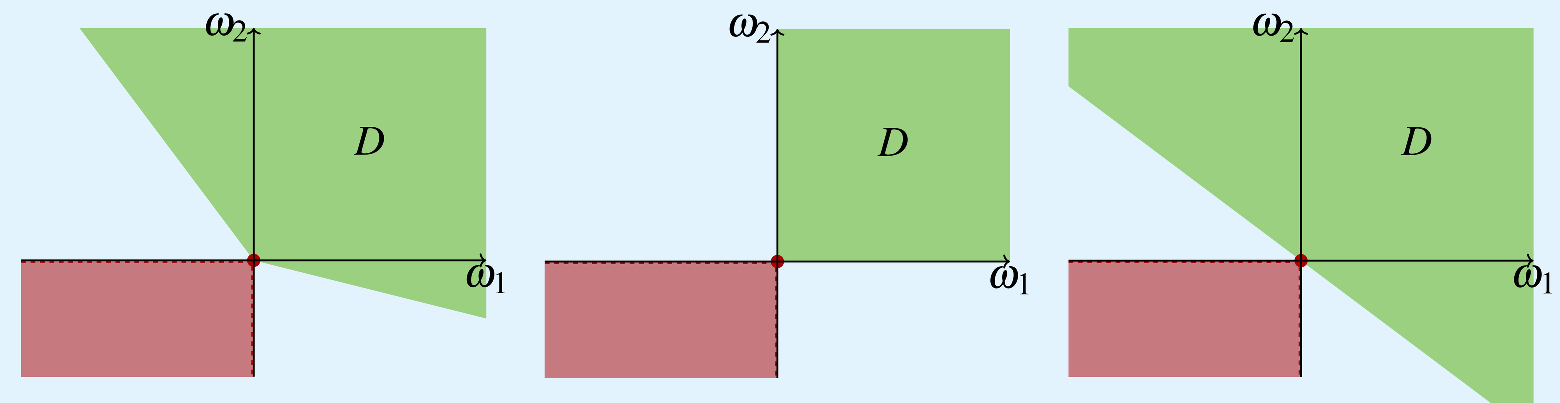
A gamble on Ω is a real-valued map on Ω. It is interpreted as an uncertain reward: if you have f then your capital changes by $f(\omega)$ when $\omega \in \Omega$ is determined.

Desirability A set of desirable gambles D is a set of gambles that the subject prefers over 0.

$f \in D$ means: the subject prefers f over 0.

Rationality axioms A set of desirable gambles D is coherent if for all gambles f and g and all real $\lambda > 0$:

- D_1. $0 \notin D$; [avoiding null gain]
- D_2. if $0 < f$ then $f \in D$; [desiring partial gain]
- D_3. if $f \in D$ then $\lambda f \in D$; [positive scaling]
- D_4. if $f, g \in D$ then $f + g \in D$. [combination]



a generic D — the vacuous D — a precise D

Conditioning Given a non-empty event $B \subseteq \Omega$, the conditional set of desirable gambles is

$$D\rfloor B = \{f \in \mathscr{L}(B) : \mathbb{I}_B f \in D\}.$$

Jeffrey’s Rule You have a coherent set of desirable gambles D on Ω, and observe a new $\check{D}$ on the partition $\mathscr{B}$: $\check{D}$ contains gambles that are constant on the elements of $\mathscr{B}$. We are looking for a coherent set of desirable gambles $\widehat{D}$ on Ω that satisfies the constraints

- $\widehat{D} \supseteq \check{D}$, [agreeing on $\mathscr{B}$]
- $\widehat{D}\rfloor B \supseteq D\rfloor B$ for all B in $\mathscr{B}$. [rigidity]

It follows from [*G. de Cooman and F. Hermans. Imprecise probability trees: Bridging two theories of imprecise probability. Artificial Intelligence, 2008*] that there is a unique smallest coherent $\widehat{D}$ that satisfies ‘agreeing on $\mathscr{B}$’ and ‘rigidity’. It is given by

$$\widehat{D} = \operatorname{posi}\Big(\check{D} \cup \textstyle\bigcup_{B\in\mathscr{B}} \mathbb{I}_B(D\rfloor B)\Big).$$

3 Sets of desirable gamble sets

$\mathscr{Q}(\Omega)$ is the collection of finite subsets of gambles on Ω. A set of desirable gamble sets $K \subseteq \mathscr{Q}$ is a collection of sets F of gambles that contain at least one gamble $f \in F$ that is preferred over 0.

$F \in K$ means: F contains at least one gamble that the subject prefers over 0.

So a set of desirable gamble sets can express more general types of uncertainty. It is equivalent to a choice function: $F \in K \Leftrightarrow 0 \notin C(\{0\} \cup F)$. [*T. Seidenfeld et al., Coherent choice functions under uncertainty. Synthese 2010*]

Rationality axioms A set of desirable gamble sets $K \subseteq \mathscr{Q}$ is coherent if for all F, F_1 and F_2 in $\mathscr{Q}$ and all $\{\lambda_{f,g}, \mu_{f,g} : f \in F_1, g \in F_2\} \subseteq \mathbb{R}$:

- K_0. $\emptyset \notin K$;
- K_1. $F \in K \Rightarrow F \setminus \{0\} \in K$;
- K_2. $\{f\} \in K$, for all f in $\mathscr{L}_{>0}$;
- K_3. if $F_1, F_2 \in K$ and if, for all f in F_1 and g in F_2, $(\lambda_{f,g}, \mu_{f,g}) > 0$, then $$\{\lambda_{f,g} f + \mu_{f,g} g : f \in F_1, g \in F_2\} \in K;$$
- K_4. if $F_1 \in K$ and $F_1 \subseteq F_2$ then $F_2 \in K$.

Here $\lambda_{1:n} := (\lambda_1, \ldots, \lambda_n) > 0$ means ‘$\lambda_k \geq 0$ for all k, and $\lambda_\ell > 0$ for at least one ℓ’.

Representation For any coherent set of desirable gambles D, let $K_D := \{F \in \mathscr{Q} : F \cap D \neq \emptyset\}$ be the set of desirable gamble sets that represents Walley–Sen maximality.
A set of desirable gamble sets K is coherent if and only if there is a non-empty representing set of coherent sets of desirable gambles $\mathbf{D}$ such that $K = \bigcap_{D \in \mathbf{D}} K_D$, and the largest such set is $\mathbf{D}(K) := \{D : K \subseteq K_D\}$.
[*J. De Bock and G. de Cooman. Interpreting, axiomatising and representing coherent choice functions in terms of desirability. ISIPTA 2019*]

Conditioning Given a non-empty event $B \subseteq \Omega$, the conditional set of desirable gamble sets is

$$K\rfloor B = \{F \in \mathscr{Q}(B) : \mathbb{I}_B F \in K\}.$$

Jeffrey’s Rule You have a coherent set of desirable gamble sets K on Ω, and observe a new $\check{K}$ on the partition $\mathscr{B}$. We are looking for a coherent set of desirable gamble sets $\widehat{K}$ on Ω that satisfies the constraints

- $\widehat{K} \supseteq \check{K}$, [agreeing on $\mathscr{B}$]
- $\widehat{K}\rfloor B \supseteq K\rfloor B$ for all B in $\mathscr{B}$. [rigidity]

There is a unique smallest coherent $\widehat{K}$ that satisfies ‘agreeing on $\mathscr{B}$’ and ‘rigidity’. It is given by

$$\widehat{K} = \operatorname{Rs}\Big(\operatorname{Posi}\Big(\check{K} \cup \textstyle\bigcup_{B\in\mathscr{B}} \mathbb{I}_B(K\rfloor B)\Big)\Big).$$

4 Example: combination of two decision rules

finite set of pmfs $\mathscr{M} \subseteq \operatorname{int}(\Sigma_\Omega)$
The agent uses **maximality**:
$K = \{F : (\exists f \in F) \min_{p\in\mathscr{M}} E_p(f) > 0\}$

finite set of pmfs $\check{\mathscr{M}} \subseteq \operatorname{int}(\Sigma_{\mathscr{B}})$
The agent uses **E-admissibility**:
$\check{K} = \{F : (\forall p \in \check{\mathscr{M}})(\exists f \in F) E_p(f) > 0\}$

(?) Can we update $\mathscr{M}$ using the new information $\check{\mathscr{M}}$, even if we use different decision rules?

Use Jeffrey’s Rule for sets of desirable gamble sets.

In general, the result $\widehat{K}$ of Jeffrey’s Rule is represented by

$$\widehat{\mathbf{D}} := \Big\{\operatorname{posi}\Big(\check{D} \cup \bigcup_{B\in\mathscr{B}} \mathbb{I}_B(D\rfloor B)\Big) : \check{D} \in \mathbf{D}(\check{K}), D \in \mathbf{D}(K)\Big\}.$$

In the present context, this representation is simplified as

$$\Big\{\operatorname{posi}\Big(D_{\check{p}} \cup \bigcup_{B\in\mathscr{B}} \mathbb{I}_B(D_{\mathscr{M}}\rfloor B)\Big) : \check{p} \in \check{\mathscr{M}}\Big\}$$

and as a consequence

$$F \in \widehat{K} \Leftrightarrow (\forall \check{p} \in \check{\mathscr{M}})(\exists f \in F) E_{\check{p}}\big(\min_{p\in\mathscr{M}} E_p(f|\mathscr{B})\big) > 0.$$

Combination of maximality and E-admissibility

5 Special cases: Jeffrey’s Rule for non-additive measures

(?) Is there a version of Jeffrey’s Rule for non-additive measures?

Consider a special class $\mathscr{C}$ of coherent lower probabilities $\underline{P}$. We lift the domain of $\underline{P}$ to gambles f: $\underline{P}(f) := \min\{E(f) : (\forall A \subseteq \Omega) E(\mathbb{I}_A) \geq \underline{P}(A)\}$.
You have a lower probability $\underline{P} \in \mathscr{C}$ on Ω, and observe a new lower probability $\check{\underline{P}} \in \mathscr{C}$ on $\mathscr{B}$. You are looking for the least informative lower probability $\widehat{\underline{P}} \in \mathscr{C}$ such that

- $\widehat{\underline{P}}(B) \geq \check{\underline{P}}(B)$, [agreeing on $\mathscr{B}$]
- $\widehat{\underline{P}}(A|B) \geq \underline{P}(A|B)$, [rigidity]

for every $A \subseteq \Omega$ and B in $\mathscr{B}$.

Proposition. *Consider $\widehat{\underline{P}} \in \mathscr{C}$. Then $\widehat{\underline{P}}$ satisfies ‘agreeing on $\mathscr{B}$’ and ‘rigidity’ iff $\widehat{\underline{P}}(f) \geq \check{\underline{P}}(\underline{P}(f|\mathscr{B}))$ for every gamble f.*

So in order to answer the question, equivalently: check whether $\check{\underline{P}}(\underline{P}(\bullet|\mathscr{B}))$ belongs to $\mathscr{C}$.

Minitive measures Assume that $\mathscr{C}$ is the class of minitive measures $\underline{P}$:

$\underline{P}(A \cap B) = \min\{\underline{P}(A), \underline{P}(B)\}$ — *minitivity on events*

$\underline{P}(\min\{f,g\}) = \min\{\underline{P}(f), \underline{P}(g)\}$ — *minitivity on gambles*

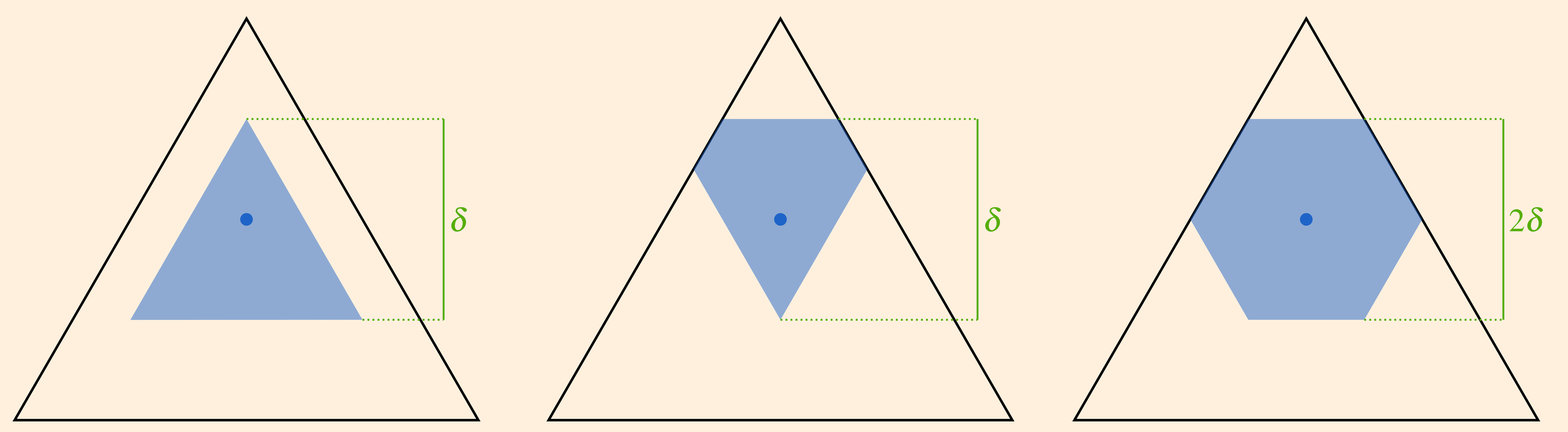
Proposition. *a) If $\underline{P}$ and $\check{\underline{P}}$ are minitive on gambles, then so is $\check{\underline{P}}(\underline{P}(\bullet|\mathscr{B}))$.*
b) If $\underline{P}$ or $\check{\underline{P}}$ is minitive on gambles, then $\check{\underline{P}}(\underline{P}(\bullet|\mathscr{B}))$ is minitive on events.
c) If $\underline{P}$ nor $\check{\underline{P}}$ is minitive on gambles, then $\check{\underline{P}}(\underline{P}(\bullet|\mathscr{B}))$ may not be minitive on events.

Distortion models Assume that $\mathscr{C}$ is either one of the classes of $\underline{P}$ that satisfy, for all $A \neq \Omega$:

$\underline{P}(A) = (1-\delta)P(A)$ — *Linear-Vacuous model*

$\overline{P}(A) = \min\{1, (1+\delta)P(A)\}$ — *Pari-Mutuel model*

$\underline{P}(A) = \max\{P(A) - \delta, 0\}$ — *Total Variation model*



Proposition. *For any of the three classes $\mathscr{C}$ of lower probabilities mentioned above: if P and $\check{P}$ belong to $\mathscr{C}$, then $\check{\underline{P}}(\underline{P}(\bullet|\mathscr{B}))$ may not belong to $\mathscr{C}$.*

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