Comonotone (natural) extension of coherent lower probabilities

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1 Setting

1.Notation

Finite ordered spaces: $\mathcal{X} = \{x_1, \ldots, x_n\}$, $\mathcal{Y} = \{y_1, \ldots, y_m\}$

Interval notation: $\tilde{a} = [\underline{a}, \overline{a}]$

Interval dominance: $\tilde{a} \preceq \tilde{b}$ if $\underline{a} \leq \underline{b}$ and $\overline{a} \leq \overline{b}$

Strict inverval dominance: $\tilde{a} \prec \tilde{b}$ if $\tilde{a} \preceq \tilde{b}$ and $\tilde{a} \neq \tilde{b}$

2.Comonotone probabilities

$\text{Supp}(P_{X,Y}) = \{(x,y) \mid P_{X,Y}(\{(x,y)\}) > 0\}$ is an increasing set in $\mathcal{X} \times \mathcal{Y}$

$\updownarrow$

$P_{X,Y}$ comonotone $\longleftrightarrow$ $X = h(Y)$, h increasing

$\updownarrow$

$F_{X,Y}(x,y) = \min\{F_X(x), F_Y(y)\}\ \forall (x,y)$

3.Comonotone lower probabilities

P comonotone $\forall P \in \mathcal{M}(\underline{P}_{X,Y})$ $\longleftrightarrow$ $\underline{P}_{X,Y}$ comonotone lower probability

$\underline{P}_{X,Y}$ comonotone lower probability $\longrightarrow$ $\text{Supp}(\underline{P}_{X,Y}) = \{(x,y) \mid \overline{P}_{X,Y}(\{(x,y)\}) > 0\}$ increasing set in $\mathbb{R}^2$

$\underline{P}_{X,Y}$ comonotone lower probability $\longrightarrow$ $\overline{F}_{X,Y}(x,y) = \min\{\overline{F}_X(x), \overline{F}_Y(y)\}$, $\underline{F}_{X,Y}(x,y) = \min\{\underline{F}_X(x), \underline{F}_Y(y)\}$

2 Aim

1.Scheme

$\underline{P}_{X,Y} \longrightarrow \underline{P}_X, \underline{P}_Y$

$\downarrow \qquad \downarrow$

$(\underline{F}_{X,Y}, \overline{F}_{X,Y}) \longrightarrow (\underline{F}_X, \overline{F}_X), (\underline{F}_X, \overline{F}_X)$

$$S = \{(x_i, y_j) \mid \widetilde{F_{X,Y}}(x_{i-1}, y_j) \prec \widetilde{F_{X,Y}}(x_i, y_j) \text{ and } \widetilde{F_{X,Y}}(x_i, y_{j-1}) \prec \widetilde{F_{X,Y}}(x_i, y_j)\}$$

2.Comonotone extension of $\underline{P}_X$ and $\underline{P}_Y$

Uncertainty about X and Y: $\underline{P}_X, \underline{P}_Y$

Def.1: $\underline{P}_{X,Y}$ is a comonotone extension of $\underline{P}_X$, $\underline{P}_Y$ if…

…$\underline{P}_{X,Y}$ is comonotone

…its marginal lower probabilities are $\underline{P}_X$, $\underline{P}_Y$

Def.2: $\underline{E}_{X,Y}$ is the comonotone natural extension of $\underline{P}_X$, $\underline{P}_Y$ if…

…$\underline{E}_{X,Y}$ is a comonotone extension

…$\underline{E}_{X,Y} \leq \underline{P}_{X,Y}$ for any other comonotone extension $\underline{P}_{X,Y}$

3.Aim

Comonotone extension of $\underline{P}_X$ and $\underline{P}_Y$…

…existence?

…construction?

…uniqueness?

Comonotone natural extension of $\underline{P}_X$ and $\underline{P}_Y$…

…existence?

…construction?

3 Existence

1.Existence

Thm.1: Given a maximal increasing set $S^* \supseteq S$, a comonotone extension $\underline{P}_{X,Y}$ of $\underline{P}_X$, $\underline{P}_Y$ satisfying $\text{Supp}(\underline{P}_{X,Y}) \subseteq S^*$ exists if and only if

1. $\widetilde{F}_X(x) \preceq \widetilde{F}_Y(y)$ or $\widetilde{F}_Y(y) \preceq \widetilde{F}_X(x)$
2. for any $A \subseteq \mathcal{X}$, there exists $P_X \in \mathcal{M}(\underline{P}_X)$ and $P_Y \in \mathcal{M}(\underline{P}_Y)$ such that $P_X(A) = \underline{P}_X(A)$ and $\text{Supp}(P_{X,Y}) \subseteq S^*$
3. for any $B \subseteq \mathcal{Y}$, there exists $P_X \in \mathcal{M}(\underline{P}_X)$ and $P_Y \in \mathcal{M}(\underline{P}_Y)$ such that $P_Y(B) = \underline{P}_Y(B)$ and $\text{Supp}(P_{X,Y}) \subseteq S^*$

2.Checking properties 2 and 3

Thm.2: Define a graph $\mathcal{G}_{S^*} = (\mathcal{V}, \mathcal{E})$ where:

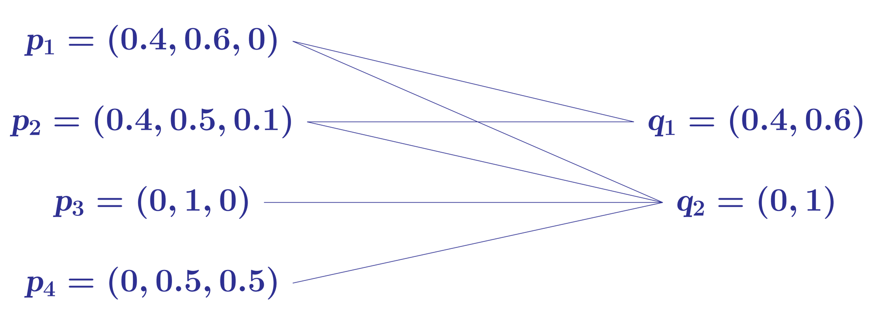
$\mathcal{V} = \text{ext}(\mathcal{M}(\underline{P}_X)) \cup \text{ext}(\mathcal{M}(\underline{P}_Y))$

$\mathcal{E} = \{(p_i, q_j) \in \text{ext}(\mathcal{M}(\underline{P}_X)) \times \text{ext}(\mathcal{M}(\underline{P}_Y)) \mid \text{Supp}(M(p_i, q_j)) \subseteq S^*\}$

Properties 2 and 3 hold if and only if $\mathcal{G}_{S^*}$ has no isolated nodes.

3.Example

$S^* = \{(x_1, y_1), (x_1, y_2), (x_2, y_2), (x_3, y_2)\} \supset S$
maximal increasing



4 Construction

A	$\underline{P}_X(A)$
$\{x_1\}$	0
$\{x_2\}$	0.5
$\{x_3\}$	0
$\{x_1, x_2\}$	0.5
$\{x_1, x_3\}$	0
$\{x_2, x_3\}$	0.6

B	$\underline{P}_Y(B)$
$\{y_1\}$	0
$\{y_2\}$	0.6

x_i	x_1	x_2	x_3
$\underline{F}_X(x_i)$	0	0.5	1
$\overline{F}_X(x_i)$	0.4	1	1

y_j	y_1	y_2
$\underline{F}_Y(y_j)$	0	1
$\overline{F}_Y(y_j)$	0.4	1

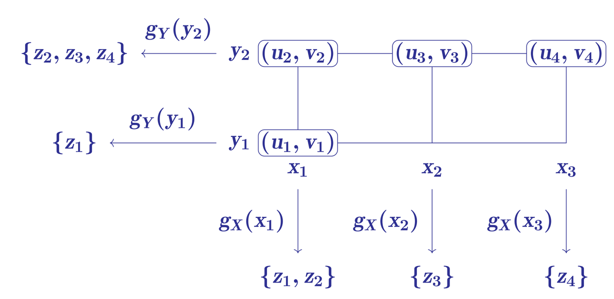
S highlighted in blue

$\widetilde{F}_X(x_i)$	$[0, 0.4]$	$[0.5, 1]$	$[1, 1]$	
y_2	$[0, 0.4]$	$[0.5, 1]$	$[1, 1]$	$[1, 1]$
y_1	$[0, 0.4]$	$[0, 0.4]$	$[0, 0.4]$	$[0, 0.4]$
$\widetilde{F}_{X,Y}(x_i, y_j)$	x_1	x_2	x_3	$\widetilde{F}_Y(y_j)$

$S^* = \{(x_1, y_1), (x_1, y_2), (x_2, y_2), (x_3, y_2)\} = \{(u_1, v_1), (u_2, v_2), (u_3, v_3), (u_4, v_4)\}$

$\underline{P}_X, \underline{P}_Y \longrightarrow (\underline{F}_X, \overline{F}_X), (\underline{F}_Y, \overline{F}_Y) \longrightarrow (\underline{F}_{X,Y}, \overline{F}_{X,Y}) \longrightarrow S^* \supseteq S$ maximal increasing

$S^* \supseteq S \xrightarrow{\;g(u_i, v_i) = z_i,\ g_X(A) = g(A \times \mathcal{Y}),\ g_Y(B) = g(\mathcal{X} \times B)\;} \mathcal{Z}$



$\mathcal{Z} \longrightarrow \underline{P}_Z : \mathcal{K} \to [0, 1]$, where $\mathcal{K} = \{g_X(A), g_Y(B) \mid A \subseteq \mathcal{X}, B \subseteq \mathcal{Y}\}$, $\underline{P}_Z(g_X(A)) = \underline{P}_X(A)$, $\underline{P}_Z(g_Y(B)) = \underline{P}_Y(B)$

$\underline{P}_Z : \mathcal{K} \to [0, 1] \xrightarrow{\text{Natural extension}} \underline{E}_Z \longrightarrow \underline{P}_{X,Y}$, with $\underline{P}_{X,Y}(C) = \underline{P}_Z(g(C \cap S^*))$

$\underline{E}_Z(D) = \min\{P_Z(D) \mid P_Z \in \mathcal{M}(\underline{P}_Z)\}$

$\underline{P}_X, \underline{P}_Y \longrightarrow \underline{P}_{X,Y}$

$D \in \mathcal{K}$	$\underline{P}_Z(D)$
$\{z_1, z_2\} = g_X(x_1)$	$\underline{P}_Z(\{z_1, z_2\}) = \underline{P}_X(\{x_1\}) = 0$
$\{z_3\} = g_X(x_2)$	$\underline{P}_Z(\{z_3\}) = \underline{P}_X(\{x_2\}) = 0.5$
$\{z_4\} = g_X(x_3)$	$\underline{P}_Z(\{z_4\}) = \underline{P}_X(\{x_3\}) = 0$
$\{z_1, z_2, z_3\} = g_X(\{x_1, x_2\})$	$\underline{P}_Z(\{z_1, z_2, z_3\}) = \underline{P}_X(\{x_1, x_2\}) = 0.5$
$\{z_1, z_2, z_4\} = g_X(\{x_1, x_3\})$	$\underline{P}_Z(\{z_1, z_2, z_4\}) = \underline{P}_X(\{x_1, x_3\}) = 0$
$\{z_3, z_4\} = g_X(\{x_2, x_3\})$	$\underline{P}_Z(\{z_3, z_4\}) = \underline{P}_X(\{x_2, x_3\}) = 0.6$
$\{z_1\} = g_Y(y_1)$	$\underline{P}_Z(\{z_1\}) = \underline{P}_Y(\{y_1\}) = 0$
$\{z_2, z_3, z_4\} = g_Y(y_2)$	$\underline{P}_Z(\{z_2, z_3, z_4\}) = \underline{P}_Y(\{y_2\}) = 0.6$
$\mathcal{Z}$	$\underline{P}_Z(\mathcal{Z}) = 1$

$\mathcal{Z} \overset{g}{\longleftrightarrow} S^*$

$z_1 \longleftrightarrow (u_1, v_1)$

$z_2 \longleftrightarrow (u_2, v_2)$

$z_3 \longleftrightarrow (u_3, v_3)$

$z_4 \longleftrightarrow (u_4, v_4)$

5 Uniqueness

1.Uniqueness

There is not uniqueness!

Example:

$S^* = \{(x_1, y_1), (x_1, y_2), (x_2, y_2), (x_3, y_2)\}$

$S' = \{(x_1, y_1), (x_2, y_1), (x_2, y_2), (x_3, y_2)\}$

give rise to different comonotone extensions.

2.Comonotone natural extension

Thm.3: Let $S_1^*, \ldots, S_k^*$ be all the maximal increasing supersets of S giving rise to the comonotone extensions $\underline{P}_1, \ldots, \underline{P}_k$. The comonotone natural extension exists if and only if

$$\cup_{i=1}^{k} \text{Supp}(\underline{P}_i)$$

is an increasing set.

Any maximal increasing superset of S determines the comonotone natural extension!

3.A necessary condition

Prop.4: If the comonotone natural extension exists, for every (x, y) one of the following properties hold:

a) $\underline{F}_X(x) \prec \overline{F}_Y(y)$

b) $\underline{F}_Y(y) \prec \overline{F}_X(x)$

c) $\underline{F}_X(x) = \overline{F}_X(x) = \underline{F}_Y(y) = \overline{F}_Y(y)$

Necessary … but not sufficient!

6 At a glance

1.Conclusions

ISIPTA'21 Comonotone extension of $(\underline{F}_X, \overline{F}_X), (\underline{F}_Y, \overline{F}_Y) \rightarrow$ existence, construction and uniqueness

ISIPTA'23 Comonotone extension of $\underline{P}_X, \underline{P}_Y \rightarrow$ existence, construction and uniqueness

- Existence: not always … ✗
- Existence: characterisation ✓
- Existence: simple way of checking the existence ✓
- Construction: simple method! ✓
- Uniqueness: no ✗
- Comonotone natural extension: Characterisation of existence & construction ✓

2.References

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