Desirable Gambles Based on Pairwise Comparisons

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Ongoing Research about a Particular Case of Sets of Desirable Gambles [1, 2, 3]

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1.- Basic Concepts

Definition

A gamble g is said to express a pairwise desirability if and only if it can be expressed as $g = I_x - bI_y$, where $b > 0$ and $x, y \in X$.

Definition

An initial specification of pairwise desirable gambles is a mapping $t : X \times X \to \mathbb{R} \cup \mathbb{R}^- \cup \{+\infty\}$, where $\mathbb{R}^- = \{a^- : a \in \mathbb{R}\}$, and $t(x, x) = 1^-$.

For any $a, b \in \mathbb{R}$, with $a < b$, we have $a^- < a < b^- < b < +\infty$. Intuitively, a^- is a number just before a (to leave open a set of gambles.

Definition

Given initial specification t, the associated set of gambles is:

$$\mathcal{T} = \{I_x - bI_y : b \in \mathbb{R}, 0 < b \le t(x, y)\} \qquad (1)$$

- If $t(x, y) = 0$, no gamble $I_x - bI_y$, with $b > 0$ is initially desirable;
- if f $t(x, y) = +\infty$, any gamble $I_x - bI_y$, with $b > 0$ is desirable;
- if $t(x, y) = a$, then any gamble $I_x - bI_y$ with $b \in (0, a]$ is desirable;
- if $t(x, y) = a^-$, then any gamble $I_x - bI_y$ with $b \in (0, a)$ is desirable.

Example

Assume that $X = \{x_1, x_2, x_3\}$ and the initial specification given by the following matrix:

$$T = \begin{pmatrix} 1^- & 0.2 & 0.3 \\ 1 & 1^- & 0.5 \\ 2 & 1 & 1^- \end{pmatrix}$$

We are specifying that $I_{x_1} - 0.2I_{x_2}, I_{x_2} - I_{x_1}$ are both desirable.

Graphical Representation

If X has three elements, the associated credal set to a set of pairwise gambles is demilited by lines pasing going from a triangle vertex to the opposite side as:

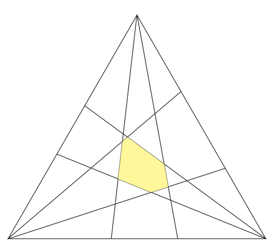


Figure: The associated credal set

Convolution

If t is a specification and i is a natural number, then, t^i is recursively defined:

$$t^1 = t, \qquad t^{i+1} = \max\{t^i \circ t, t^i\}$$

where $t_1 \circ t_2(x, y) = \sup\{t_1(x, z).t_2(z, y) : z \in X\}$

Result

If $0 < b \le t^i(x, y)$, there are $m \le i$ gambles $I_{x_1} - b_1I_{y_1}, \ldots, I_{x_m} - b_mI_{y_m} \in \mathcal{T}$ and positive numbers $\alpha_i, \ldots, \alpha_b$ such that

$$I_x - bI_y \ge \alpha_1(I_{x_1} - b_1I_{y_1}) + \cdots + \alpha_m(I_{x_m} - b_mI_{y_m})$$

Result

$$\{t^i\}_{i \ge 1} \uparrow \bar{t}$$

The natural Extension

Example

Assuming the initial representation of Example 1, matrix T converges in one iteration to its limit:

$$\bar{T} = \begin{pmatrix} 1^- & \mathbf{0.3} & 0.3 \\ 1 & 1^- & 0.5 \\ 2 & 1 & 1^- \end{pmatrix}$$

2. Natural Extension

Result

If t is an initial specification, then its associated set of gambles $\mathcal{T}$ avoids sure loss if and only if $\bar{t}(x, x) < 1, \forall x \in X$.

Finite Case

If X is finite, and we consider a weighted graph with a node for each $x \in X$ and a weight $t(x, y)$ for a link from x to y, and define the value of a path as the product of the weights of its links, then $t^i(x, y)$ is the maximum weight of the paths going from x to y.

$$\bar{t}(x, x) \ge 1 \Leftrightarrow t^n(x, x) \ge 1$$

If $\bar{t}(x, x) < 1, \forall x$, then $\bar{t}(x, y) = t^n(x, y)$.

Result

If t is an initial specification such that $t(x, x) < 1$ for any $x \in X$ and $\bar{\mathcal{T}}$ is the natural extension of $\mathcal{T}$, then a pairwise gamble $I_x - bI_y \in \bar{\mathcal{T}}$, if and only if $b \le \bar{t}(x, y)$.

Computing Gambles in the Natural Extension

The problem we consider now is when a generic gamble g is in the natural extension.

$$P_g = \{x \in X : g(x) > 0\}, \quad N_g = \{x \in X : g(x) < 0\}.$$

This problem can be rephrased as a max flow problem with gain/loss factors [6]. In particular, the problem is as follows:

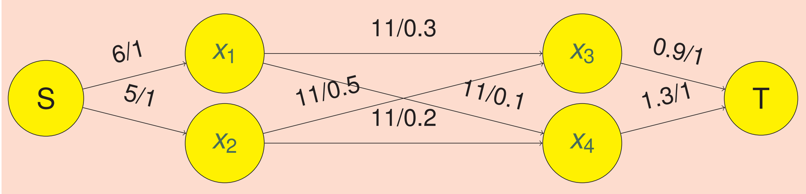
- There is a source node s and a sink node t.
- There are a node for each $x \in P_g$ and a node for each $y \in N_g$
- There is a link from s to each node $x \in P_g$ with capacity 1 and gain factor of 1.
- There is a link from each node $y \in N_g$ and t with capacity 1 and gain factor of 1.
- There is a link from each node $x \in P_g$ to each node $y \in N_g$ with unlimited capacity (it could be set to the value $\sum_{x \in P_g} g(x)$ and gain factor of $\bar{t}(x, y)$.

Example

Assume that $X = \{x_1, x_2, x_3, x_4\}$ and pairwise natural extension given by matrix:

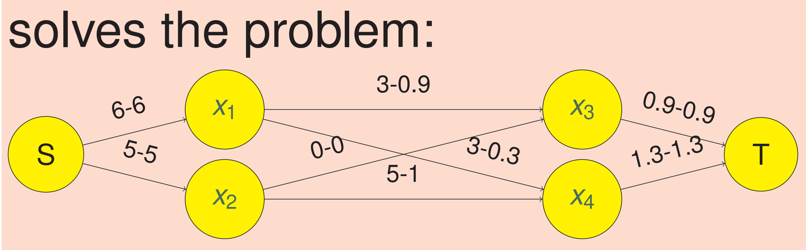
$$T = \begin{pmatrix} 1^- & 0.3 & 0.3 & 0.1 \\ 1 & 1^- & 0.5 & 0.2 \\ 2 & 1 & 1^- & 0.3 \\ 1 & 2 & 1 & 1^- \end{pmatrix}$$

The max flow problem associated with gamble $6I_{x_1} + 5I_{x_2} - 0.9I_{X3} - 1.3I_{x_4}$ is represented in the following graph:



g is in the natural extension if and only if there is a flux in which an amount of $\sum_{y \in N_g} g(y)$ arrives to t. In that case the coefficient $a_{x,}$ is equal to the flux entering the link from x to y.

A flux as in the figure (where $a - b$ means that a units enter the link and b units arrive to the link end) solves the problem:



It corresponds to the expression:

$6\, I_{x_1} + 5I_{x_2} - 0.9IX_3 - 1.3I_{x_4} \ge 3(I_{x_1} - 0.3I_{x_2}) + 3(I_{x_1} - 0.1I_{x_4}) + 5(I_{x_2} - 0.1I_{x_4})$

3.- Equiprobability

Equiprobability is represented by the initial representation $\bar{t}(x, y) = 1^-$, $\forall x, y \in X$. A gamble $I_x - aI_y$ is desirable whenever $0 < a < 1$.

Result

If X is finite, a gamble g is desirable if and only if such that $\sum_{x \in X} g(x) > 0$.

The associated credal set contains only one element: P_u, the uniform probability in X.

Result

If X is infinite, a gamble g is desirable when N_g is finite and there is $H \subseteq P_g$ with H finite and $\sum_{x \in (H \cup N_g)} g(x) > 0$.

Result

If X is infinite, then credal set associated to $\bar{t}$ is equal to the set of all the finitely additive probability measures in X such that $P(H) = 0$ for any $H \subseteq X$ finite.

Result

If H is finite, the conditioning of $\bar{t}$ to H, is the finite uniform.

Discounting

1. If $\mathcal{D}$ is a set of desirable gambles and $\epsilon \in [0, 1]$, then the discounting of $\mathcal{D}$ by ϵ is

$$\mathcal{D}^\epsilon = \{g - \epsilon \inf(g) I_{N_g \cup P_g} : g \in \mathcal{D}\} \setminus \{0\}.$$

If $\mathcal{M}$ is the credal set associated with $\mathcal{D}$, the credal set associated with $\mathcal{D}^\epsilon$ is

$\mathcal{M}_\epsilon = (1 - \epsilon)\mathcal{M} + \epsilon \mathcal{M}_0 = \{(1 - \epsilon)P + \epsilon Q : P \in M_{\mathcal{D}}, Q \in \mathcal{M}_0\}$

$\mathcal{M}_0$ is the vacuous credal set.

With Pairwise specifications:

$$t^\epsilon(x, y) = t(x, y)\frac{1 - \epsilon}{1 + \epsilon t(x, y)}, \text{ when } x \ne y$$

However, this is not equivalent to discount the full natural extension.

2. Second discounting:

$$\mathcal{D}^\epsilon = \{g - \epsilon(\sup - \inf(g)) I_{N_g \cup P_g} : g \in \mathcal{D}\} \setminus \{0\}.$$

Associated to distance

$D(p, q) = 1/2 \sum_x |p(x) - q(x)|$

$$t^\epsilon(x, y) = \max\{0, \frac{t(x, y) - \epsilon t(x, y) - \epsilon}{1 + \epsilon t(x, y) + \epsilon}, \text{ when } x \ne y\}$$

Multiplicative Preference Relationships

- A multiplicative preference relationship [4] is a matrix A with values $a_{i,j}.a_{j,i} = 1$.
  There are several definitions of consistency, but one of them is $a_{i,j}a_{j,k} = a_{i,k}$.
  It corresponds to maximal pairwise specifications.
- In [5] generalized to intervals. Consistency:

$$\{(p_1, \ldots, p_n) : l_{i,j} \le p_i/p_j \le u_{i,j}, \sum_i p_i = 1, p_i > 0\} \ne \emptyset$$

Very similar to a pairwise specification with $l_{i,j} = t(x_i, x_j), u_{i,j} = 1/t(x_j, x_i)$

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