Expected Time Averages in Markovian Imprecise Jump Processes: A Graph-Theoretic Characterisation of Weak Ergodicity

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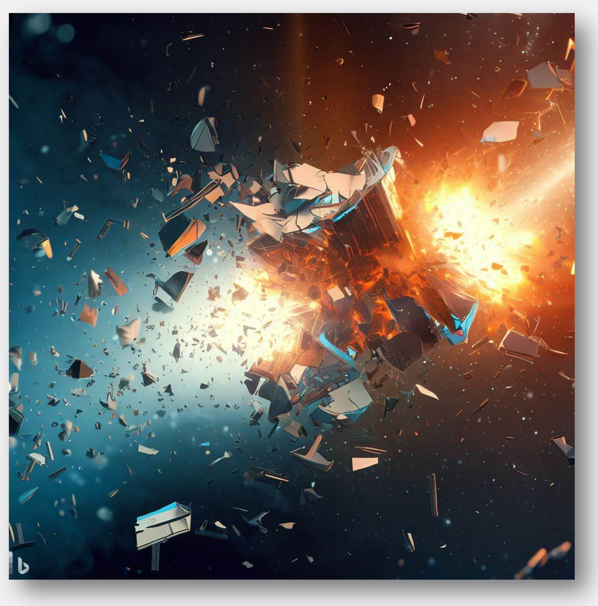
Context and Motivation

- An orbital collision typically produces approximately 10,000 fragments of space debris, each posing a significant threat to other operational satellites.
- Space agencies employ the Maximum Probability metric to assess spacecraft collision risks; however, its validity is limited to short-term events.
- Need a robust model adapted to assess long-term collision risks induced by an orbital breakup.
  - High fidelity propagation model
  - Maximum number of collisions over future decades



Objectives

- Crucial to:
  - Use **Continuous-time Imprecise Markov Chain** to model the cloud of space debris together with high fidelity orbital propagation models.
  - Use **Upper expected time averages** to express maximum number of collisions.
  - **Weak Ergodicity** is about convergence of Upper expected time averages, and we need a Graph-theoretic characterization of this property.
- Solutions:
  - Definition of a Graph by the upper rate operator associated to the imprecise Markov chain, with the notion of **absorbing top class**.
  - An imprecise Markov chain is Weak Ergodic if and only if its graph has an absorbing top class.
  - Graph conditions converted into orbital model conditions

Imprecise Markov Chains

Precise Markov Chain

A continuous-time *Markov Chain* is a stochastic process P such that for all $t_1 < \cdots < t_n$ and $t \in \mathbb{R}_{>0}$,

$$E_P[f(X_{t_n+t})|X_{t_1} = x_1, \dots, X_{t_n} = x_n] = E_P[f(X_t)|X_0 = x] =: T_t[f](x),$$

where T_t is the transition operator of P. It is well known that a Markov chain is uniquely defined by its transition rate matrix Q, which satisfies the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} T_t = Q T_t, \quad T_0 = I.$$

Imprecise Markov Chain

The sensitivity of the system X_t may impose to consider that the transition rate matrix is partially known, but belongs to some set $\mathcal{Q}$ of rate matrices. An *imprecise Markov chain* $\mathcal{P}$ is a set of Markov chains P with rate matrix in $\mathcal{Q}$. Defined the upper rate operator $\overline{Q}$ by $\overline{Q}[f](x) := \sup_{Q \in \mathcal{Q}} \{Q[f](x)\}$ and the upper transition operator $\overline{T}$ by

$$\overline{T}_t[f](x) := \overline{E}[f(X_t)|X_0 = x] := \sup_{P \in \mathcal{P}} E_P[f(X_t)|X_0 = x].$$

Remarkably, they satisfy the following (non-linear) differential equation :

$$\frac{\mathrm{d}}{\mathrm{d}t} \overline{T}_t = \overline{Q}\, \overline{T}_t, \text{ with } \overline{T}_0 = I.$$

The Upper Expectation of Interest

An imprecise jump process $\mathcal{P}$ is said to be *weakly ergodic* if for all $f : \mathcal{X} \to \mathbb{R}$ and $x \in \mathcal{X}$,

$$\lim_{t \to +\infty} \overline{E}_{\mathcal{P}} \left(\frac{1}{t} \int_0^t f(X_\tau)\, \mathrm{d}\tau \,\middle|\, X_0 = x \right)$$

exists and is the same for all $x \in \mathcal{X}$.

Result & Analysis

The Graph of the imprecise Markov Chain

We let $\mathcal{G}$ be the directed graph with directed edges

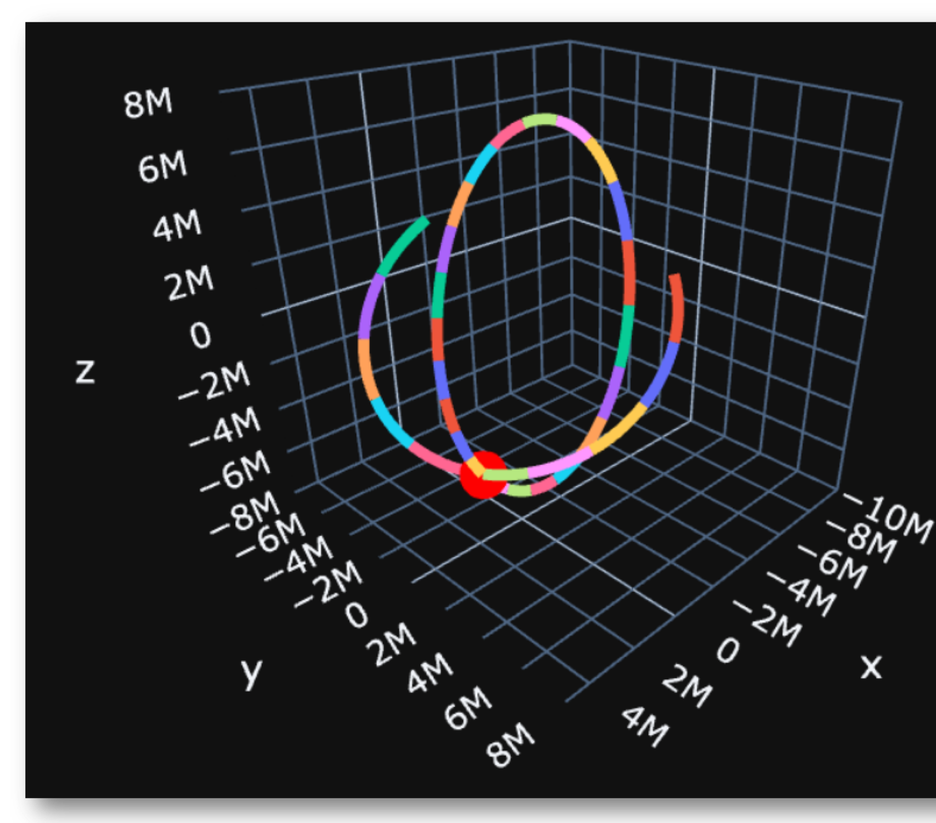
$$\mathcal{E} := \{x \to y \mid \overline{Q}[\mathbb{I}_y](x) > 0\}.$$

If the set defined by

$$\{x \in \mathcal{X} \mid (\forall y \in \mathcal{X}),\ y \to x_1 \to \cdots \to x_n \to x\}$$

is non-empty, then it is unique and called the *Top Class* of the graph $\mathcal{G}$. A top class $\mathcal{S}$ is said to be *absorbing* if for any $x \in \mathcal{X} \setminus \mathcal{S}$, $x \in A_K$ where $(A_0, \dots, A_K)$ is the sequence defined by the initial condition $A_0 := \mathcal{S}$ and, for all $k \in \{0, \dots, K-1\}$, by the recursive relation

$$A_{k+1} := A_k \cup \{x \in \mathcal{X} \setminus A_k : \overline{Q}[-\mathbb{I}_{A_k}](x) < 0\}.$$



Main Result

Consider an imprecise jump process $\mathcal{P}$, and its graph $\mathcal{G}$. Then, the two assertions are equivalent:

(i) for any $f : \mathcal{X} \to \mathbb{R}$, the limit

$$\lim_{t \to +\infty} \sup_{P \in \mathcal{P}} E_P \left(\frac{1}{t} \int_0^t f(X_\tau)\, \mathrm{d}\tau \,\middle|\, X_0 = x \right)$$

exists and is the same for all $x \in \mathcal{X}$.

(ii) The graph $\mathcal{G}$ has an absorbing top-class.

Comparison with Simulation Results

Maximum Expected collision number between the cloud of debris and different Satellites