Imprecision in Martingale-Theoretic Prequential Randomness

Consider a fair coin—so with probability 1/2 for heads—and assume that we toss it an infinite number of times. For this classical set-up, you are familiar with statements as 'The set of sequences that satisfy the law of large numbers has measure one (w.r.t. the obvious induced measure)'. You are probably less acquainted with examining the properties of a single sequence that is generated by tossing this coin. In Algorithmic Randomness, we study what it means for such infinite sequences to be 'random' w.r.t. 1/2. When adopting a martingale-theoretic approach, a sequence is then basically considered to be random if you cannot get arbitrarily rich by betting on the successive outcomes.

Classical martingale-theoretic notions of randomness not only allow for defining an infinite sequence's randomness w.r.t. probability 1/2, but also w.r.t. precise forecasting systems, which are maps from binary strings to probabilities. However, it is only rather recently that we succeeded at allowing for interval-valued forecasting systems in such notions of randomness. By doing so, we can test whether an infinite binary sequence is random w.r.t. a set of so-called compatible precise forecasting systems.

It is not always natural to define an infinite sequence's randomness w.r.t. an (imprecise) forecasting system. Consider for example a weather forecaster who provides a daily probability interval for rain in the next 24 hours. He typically only specifies forecasts 'on the fly', that is, based on the rain history he has actually observed, and doesn't provide forecasts for all rain histories that might have been and might be. As a way out of this conundrum, we adopt Dawid and Vovk's *prequential forecasting* framework, which allows us to define what it means for an infinite sequence $(I_1, x_1, I_2, x_2, \dots)$ of successive rational interval forecasts I_k and subsequent binary outcomes x_k to be random.

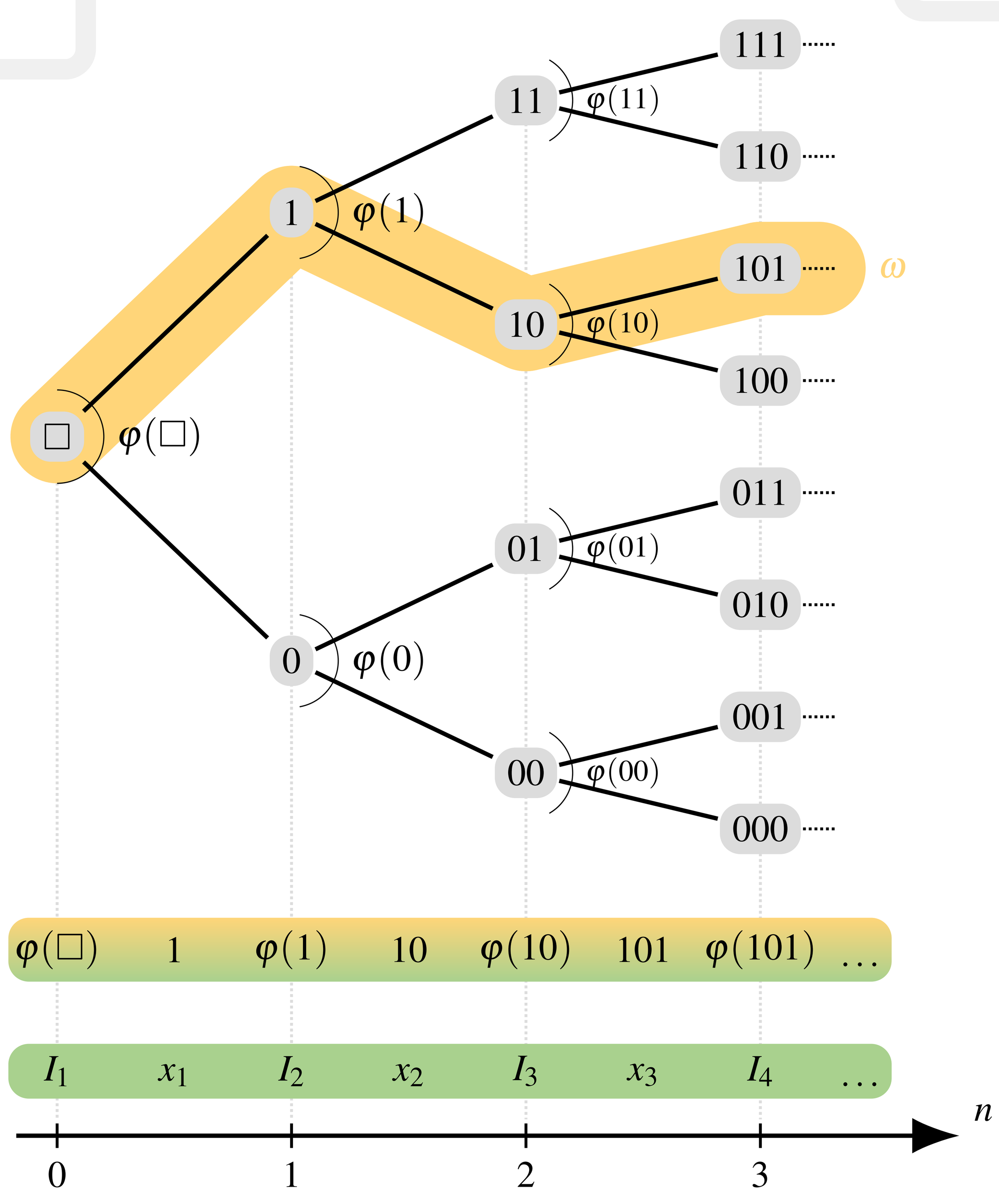
interval forecast(s)	$I \in \mathscr{I} := \{[p,q] \subseteq [0,1] : p \leq q \in \mathbb{R}\}$
situation(s)	$w \in \{0,1\}^* := \bigcup_{n \in \mathbb{N}_0} \{0,1\}^n$
path(s)	$\omega \in \{0,1\}^{\mathbb{N}}$
(rat.) forecasting system	$\varphi_{(r)} : \{0,1\}^* \to \mathscr{I}_{(r)}$

non-degenerate if $(\forall w \in \{0,1\}^*)\varphi_{(r)}(w) \not\subseteq \{0,1\}$

compatible if $(\forall n \in \mathbb{N}_0)\varphi_r(\omega_1, \dots, \omega_n) = I_{n+1}$

rat. interval forecast(s)	$I_r \in \mathscr{I}_r := \{[p,q] \subseteq [0,1] : p \leq q \in \mathbb{Q}\}$
preq. situation(s)	$(I_1, \omega_1, \dots, I_n, \omega_n) \in (\mathscr{I}_r \times \{0,1\})^*$
preq. path(s)	$(I_1, \omega_1, I_2, \omega_2, \dots) \in (\mathscr{I}_r \times \{0,1\})^{\mathbb{N}}$

non-degenerate if $(\forall n \in \mathbb{N}) I_n \neq \{1 - \omega_n\}$



Martin-Löf randomness

Betting strategies

A real-valued map $T : \{0,1\}^* \to \mathbb{R}$ is called a **test supermartingale** for a forecasting system φ if $T(\square) = 1$, $T(w) \geq 0$ and $\overline{E}_{\varphi(w)}(T(w\cdot)) \leq T(w)$ for all $w \in \{0,1\}^*$.

Definition

A path $\omega \in \{0,1\}^{\mathbb{N}}$ is **Martin-Löf random** for a forecasting system φ if $\limsup_{n \to +\infty} T(\omega_1, \dots, \omega_n) < +\infty$ for all lower semicomputable test supermartingales T for φ.

Prequential randomness

Betting strategies

A real-valued map $F : (\mathscr{I}_r \times \{0,1\})^* \to \mathbb{R}$ is called a **test superfarthingale** if $F(\square) = 1$, $F(I_1, \omega_1, \dots, I_n, \omega_n) \geq 0$ and $\overline{E}_{I_r}(F(I_1, \omega_1, \dots, I_n, \omega_n, I_r, \cdot)) \leq F(I_1, \omega_1, \dots, I_n, \omega_n)$ for all $(I_1, \omega_1, \dots, I_n, \omega_n) \in (\mathscr{I}_r \times \{0,1\})^*$ and $I_r \in \mathscr{I}_r$.

Definition

We call a sequence $(I_1, \omega_1, I_2, \omega_2, \dots) \in (\mathscr{I}_r \times \{0,1\})^{\mathbb{N}}$ of rational forecasts and outcomes **game-random** if it's non-degenerate and if all lower semicomputable test superfarthingales F satisfy $\limsup_{n \to +\infty} F(I_1, \omega_1, \dots, I_n, \omega_n) < +\infty$.

Rational restriction

For every non-degenerate computable forecasting system φ there's a recursive rational forecasting system φ_r, with $\varphi \subseteq \varphi_r$, such that a path $\omega \in \{0,1\}^{\mathbb{N}}$ is Martin-Löf random for φ if and only if it's Martin-Löf random for φ_r.

Martin-Löf '=' Prequential

Consider any non-degenerate recursive rational forecasting system φ_r. Then any path $\omega \in \{0,1\}^{\mathbb{N}}$ is Martin-Löf random for φ_r if and only if the prequential path $(\varphi_r(\square), \omega_1, \varphi_r(\omega_1), \omega_2, \dots) \in (\mathscr{I}_r \times \{0,1\})^{\mathbb{N}}$ is game-random.

Martin-Löf ⊆ Prequential

Consider any infinite sequence of interval forecasts and outcomes $(I_1, \omega_1, I_2, \omega_2, \dots) \in (\mathscr{I}_r \times \{0,1\})^{\mathbb{N}}$ that's game-random. Then the infinite sequence of outcomes ω is Martin-Löf random for any rational forecasting system φ_r that's compatible with $(I_1, \omega_1, I_2, \omega_2, \dots)$.

Property 1: universal supermartingale

For every non-degenerate computable forecasting system φ, there's a so-called *universal* lower semicomputable test supermartingale T such that any path $\omega \in \{0,1\}^{\mathbb{N}}$ is Martin-Löf random for φ if and only if $\lim_{n \to +\infty} T(\omega_1, \dots, \omega_n) < +\infty$.

Property 2: at least one random path

For every forecasting system φ, there is at least one path $\omega \in \{0,1\}^{\mathbb{N}}$ that is Martin-Löf random for φ.

Property 3: monotonicity

Consider any forecasting system φ. If a path $\omega \in \{0,1\}^{\mathbb{N}}$ is Martin-Löf random for φ, then it is Martin-Löf random as well for any forecasting system φ' for which $\varphi \subseteq \varphi'$.

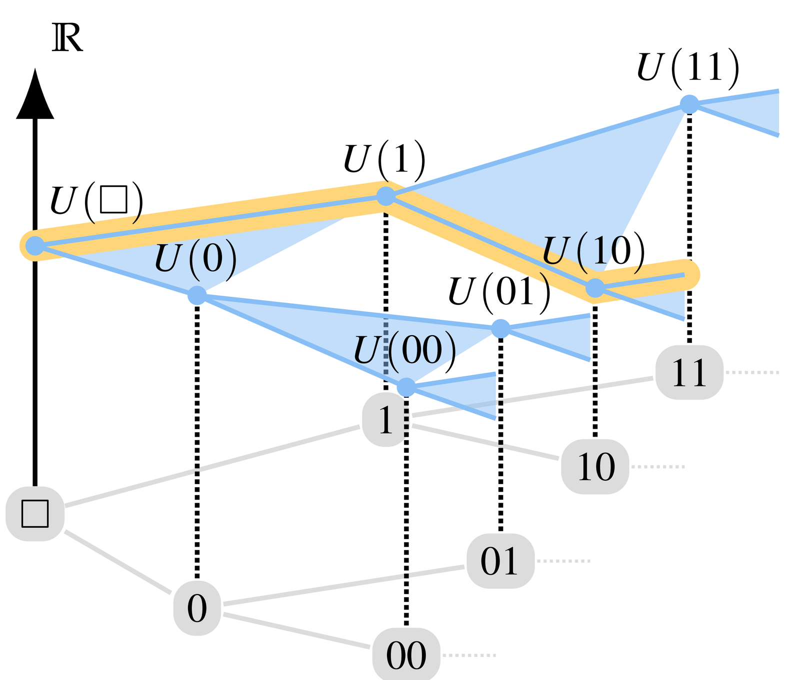
Property 4: relative frequencies

Consider any path $\omega \in \{0,1\}^{\mathbb{N}}$, any computable forecasting system φ and any recursive selection process $S : \{0,1\}^* \to \{0,1\}$ such that $\lim_{n \to +\infty} \sum_{k=0}^{n-1} S(\omega_1, \dots, \omega_k) = +\infty$. If ω is Martin-Löf random for φ, then

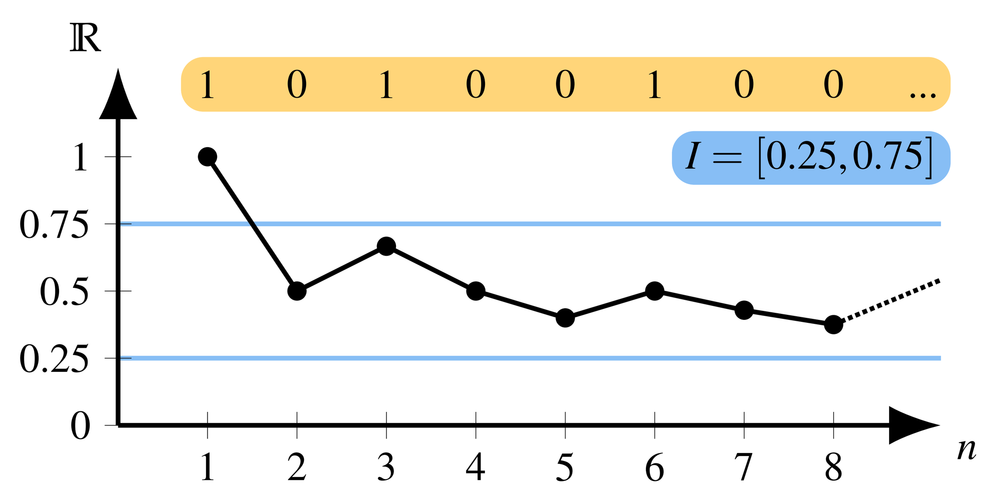
$$\liminf_{n \to +\infty} \frac{\sum_{k=0}^{n-1} S(\omega_1, \dots, \omega_k)[\omega_{k+1} - \underline{\varphi}(\omega_1, \dots, \omega_k)]}{\sum_{k=0}^{n-1} S(\omega_1, \dots, \omega_k)} \geq 0$$

and

$$\limsup_{n \to +\infty} \frac{\sum_{k=0}^{n-1} S(\omega_1, \dots, \omega_k)[\omega_{k+1} - \overline{\varphi}(\omega_1, \dots, \omega_k)]}{\sum_{k=0}^{n-1} S(\omega_1, \dots, \omega_k)} \leq 0.$$



What kind of forecasting framework do we borrow from Dawid and Vovk?



Floris Persiau & Gert de Cooman

Property 1: universal superfarthingale

There's a so-called *universal* lower semicomputable test superfarthingale U with the property that any non-degenerate prequential path $(I_1, \omega_1, I_2, \omega_2, \dots) \in (\mathscr{I}_r \times \{0,1\})^{\mathbb{N}}$ is game-random if and only if $\limsup_{n \to +\infty} U(I_1, \omega_1, \dots, I_n, \omega_n) < +\infty$.

Property 2: at least one random preq. path

For every infinite sequence of rational interval forecasts $(I_1, I_2, \dots) \in \mathscr{I}_r^{\mathbb{N}}$ there's at least one path $\omega \in \{0,1\}^{\mathbb{N}}$ such that $(I_1, \omega_1, I_2, \omega_2, \dots) \in (\mathscr{I}_r \times \{0,1\})^{\mathbb{N}}$ is game-random.

Property 3: monotonicity

Consider any recursive rational forecasting system φ_r and any game-random prequential path $(I_1, \omega_1, I_2, \omega_2, \dots) \in (\mathscr{I}_r \times \{0,1\})^{\mathbb{N}}$. If $I_{n+1} \subseteq \varphi_r(\omega_1, \dots, \omega_n)$ for all $n \in \mathbb{N}_0$, then $(\varphi_r(\square), \omega_1, \varphi_r(\omega_1), \omega_2, \dots) \in (\mathscr{I}_r \times \{0,1\})^{\mathbb{N}}$ is game-random as well.

Property 4: relative frequencies

Consider any infinite sequence of rational interval forecasts and outcomes $(I_1, \omega_1, I_2, \omega_2, \dots) \in (\mathscr{I}_r \times \{0,1\})^{\mathbb{N}}$ and any recursive *selection function* $S : (\mathscr{I}_r \times \{0,1\})^* \times \mathscr{I}_r \to \{0,1\}$ such that $\sum_{k=0}^{+\infty} S(I_1, \omega_1, \dots, I_k, \omega_k, I_{k+1}) = +\infty$. If $(I_1, \omega_1, I_2, \omega_2, \dots)$ is game-random, then

$$\liminf_{n \to +\infty} \frac{\sum_{k=0}^{n-1} S(I_1, \omega_1, \dots, I_k, \omega_k, I_{k+1})[\omega_{k+1} - \min I_{k+1}]}{\sum_{k=0}^{n-1} S(I_1, \omega_1, \dots, I_k, \omega_k, I_{k+1})} \geq 0$$

and

$$\limsup_{n \to +\infty} \frac{\sum_{k=0}^{n-1} S(I_1, \omega_1, \dots, I_k, \omega_k, I_{k+1})[\omega_{k+1} - \max I_{k+1}]}{\sum_{k=0}^{n-1} S(I_1, \omega_1, \dots, I_k, \omega_k, I_{k+1})} \leq 0.$$