No-Arbitrage Pricing with α-DS Mixtures in a Market with Bid-Ask Spreads

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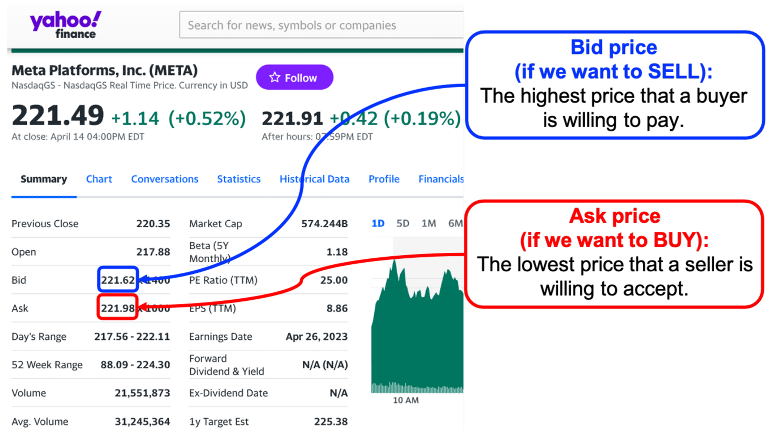
Motivation: pricing in a market with frictions

- Classical no-arbitrage pricing theory assumes that the market is **competitive** and **frictionless**
- Prices can be expressed as **discounted expectations** with respect to an "artificial" probability measure Q
- PROBLEM: Markets show frictions, mostly in the form of **bid-ask spreads**
- AIM: Replace Q with a non-additive measure so as to consider bid-ask spreads



α-DS mixtures

Consider:

- $\Omega = \{1, \ldots, n\}$ with $n \geq 1$, a finite set of states of the world
- $\mathcal{P}(\Omega)$, power set of events
- $\mathbb{R}^\Omega$, set of all random variables

Definition (α-DS mixture)

Let $\alpha \in [0,1]$. A mapping $\varphi_\alpha : \mathcal{P}(\Omega) \to [0,1]$ is called an **α-DS mixture** if there exists a belief function $Bel : \mathcal{P}(\Omega) \to [0,1]$ with dual plausibility function Pl such that, for all $A \in \mathcal{P}(\Omega)$,

$$\varphi_\alpha(A) = \alpha Bel(A) + (1-\alpha)Pl(A) = \alpha Bel(A) + (1-\alpha)(1 - Bel(A^c)).$$

The belief function Bel is said to **represent** the α-DS mixture φ_α.

We further distinguish the subclasses of **additive** and **consonant** α-DS mixtures.

Proposition (unique representation)

Let $\alpha \in [0,1]$ with $\alpha \neq \frac{1}{2}$, and $\varphi_\alpha : \mathcal{P}(\Omega) \to [0,1]$ be an α-DS mixture. Let Bel, Bel' be belief functions on $\mathcal{P}(\Omega)$. If both Bel and Bel' represent φ_α, then $Bel = Bel'$.

Properties of α-DS mixtures

Proposition (properties of a φ_α)

Let $\alpha \in [0,1]$. An α-DS mixture $\varphi_\alpha : \mathcal{P}(\Omega) \to [0,1]$ satisfies the following properties:

(i) $\varphi_\alpha(\emptyset) = 0$ and $\varphi_\alpha(\Omega) = 1$;
(ii) $\varphi_\alpha(A) \leq \varphi_\alpha(B)$, when $A \subseteq B$ and $A, B \in \mathcal{P}(\Omega)$;
(iii) φ_α is self-dual if and only if it is additive or $\alpha = \frac{1}{2}$;
(iv) φ_α is sub-additive if it is additive or $\alpha \in [0, \frac{1}{2}]$.

For every $\alpha \in [0,1]$, the class $\mathbf{M}_\alpha$ of all α-DS mixtures on $\mathcal{P}(\Omega)$ is **convex** and contains the class $\mathbf{P}$ of all probability measures on $\mathcal{P}(\Omega)$.

α-DS mixture Choquet expectation

Every φ_α uniquely extends to a functional $\mathbb{C}_{\varphi_\alpha} : \mathbb{R}^\Omega \to \mathbb{R}$ by setting, for every $X \in \mathbb{R}^\Omega$,

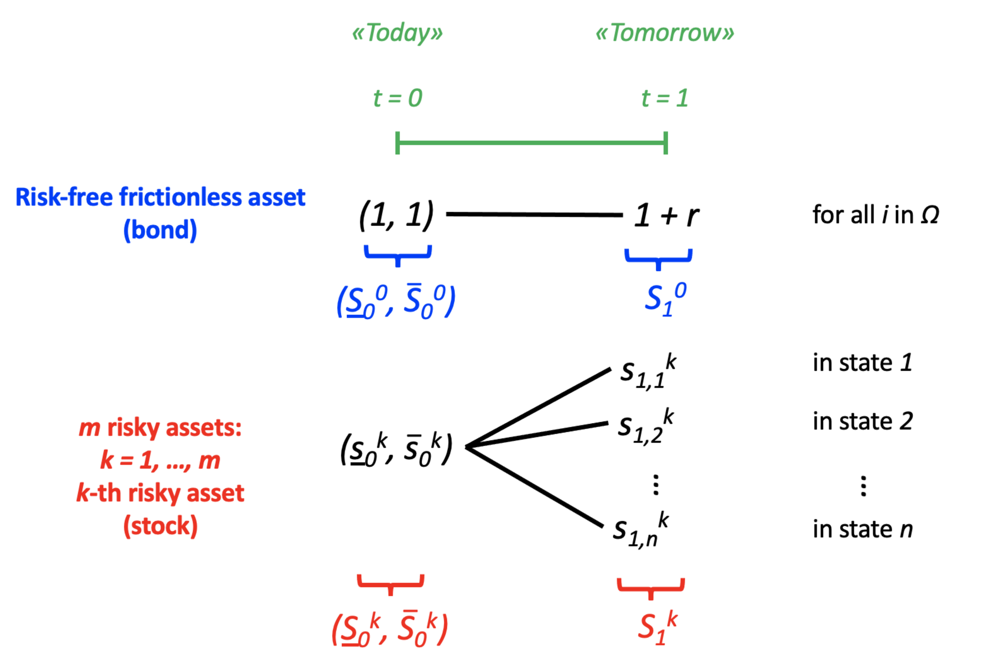
$$\mathbb{C}_{\varphi_\alpha}[X] = \oint X \, d\varphi_\alpha$$

Hurwicz-like representation: $\mathbb{C}_{\varphi_\alpha}[X] = \alpha \min_{P \in \mathcal{C}_{Bel}} \mathbb{E}_P[X] + (1-\alpha) \max_{P \in \mathcal{C}_{Bel}} \mathbb{E}_P[X]$ where $\mathcal{C}_{Bel}$ is the core of Bel

Möbius-like representation: $\mathbb{C}_{\varphi_\alpha}[X] = \sum_{B \in \mathcal{U}} [\![X]\!]^\alpha(B)\mu(B)$ where μ is the Möbius inverse of Bel and

$$\mathcal{U} = \mathcal{P}(\Omega) \setminus \{\emptyset\} \quad \text{and} \quad [\![X]\!]^\alpha : \mathcal{U} \to \mathbb{R} \quad \text{with} \quad [\![X]\!]^\alpha(B) = \alpha \min_{i \in B} X(i) + (1-\alpha) \max_{i \in B} X(i)$$

One-period market with bid-ask spreads



α-Mixture prices at time $t = 0$:

$M_0 = (S_0^0, S_0^1, \ldots, S_0^m)^T$

$S_0^k = \alpha \underline{S}_0^k + (1-\alpha)\overline{S}_0^k$

Payoffs at time $t = 1$:

$M_1 = (S_1^0, S_1^1, \ldots, S_1^m)^T$

No-arbitrage pricing under α-PRU

Given a portfolio $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_m)^T \in \mathbb{R}^{m+1}$ we define:

Price at time $t = 0$: $V_0^\lambda = \lambda_0 + \sum_{k=1}^m \lambda_k S_0^k$

Payoff under α-PRU at time $t = 1$: $V_1^\lambda = \lambda_0(1+r) + \sum_{k=1}^m \lambda_k [\![S_1^k]\!]^\alpha$

α-PRU principle at time $t = 1$

PRU (Partially Resolving Uncertainty): An agent may only acquire that $B \neq \emptyset$ occurs, without knowing which is the true $i \in B$

α-pessimism: An agent always considers the α-mixture between the minimum and the maximum of random payoffs on every $B \neq \emptyset$

Theorem (First FTAP under α-PRU)

Let $\alpha \in [0,1]$. The following conditions are equivalent:

(i) there exists an α-DS mixture $\widehat{\varphi_\alpha}$ represented by a belief function strictly positive on $\mathcal{U}$ and such that $\frac{\mathbb{C}_{\widehat{\varphi_\alpha}}[S_1^k]}{1+r} = S_0^k$, for $k = 1, \ldots, m$;

(ii) for every $\lambda \in \mathbb{R}^{m+1}$ none of the following conditions holds:

(a) $V_1^\lambda(\{i\}) = 0$, for $i = 1, \ldots, n$, $V_1^\lambda(B) \geq 0$, for all $B \in \mathcal{U} \setminus \{\{i\} : i \in \Omega\}$ and $V_0^\lambda < 0$;
(b) $V_1^\lambda(\{i\}) \geq 0$, for $i = 1, \ldots, n$, not all 0, $V_1^\lambda(B) \geq 0$, for all $B \in \mathcal{U} \setminus \{\{i\} : i \in \Omega\}$, and $V_0^\lambda \leq 0$.

Theorem (Second FTAP under α-PRU)

Let $\alpha \in [0,1]$. If the market satisfies condition *(ii)* of the First FTAP under α-PRU and is **α-PRU complete**, i.e., for $\mathcal{U} = \{B_1, \ldots, B_{2^n-1}\}$, it is $m \geq 2^n - 1$ and $S_1^k = \mathbf{1}_{B_k}$, for $k = 1, \ldots, 2^n - 1$, then the α-DS mixture $\widehat{\varphi_\alpha}$ in condition *(i)* of the First FTAP under α-PRU is unique.

α-DS mixture no-arbitrage price of a payoff $X_1 \in \mathbb{R}^\Omega$

$$X_0 = (1+r)^{-1}\mathbb{C}_{\widehat{\varphi_\alpha}}[X_1] = (1+r)^{-1}\left(\alpha \min_{Q \in \mathcal{C}_{\widehat{Bel}}} \mathbb{E}_Q[X_1] + (1-\alpha) \max_{Q \in \mathcal{C}_{\widehat{Bel}}} \mathbb{E}_Q[X_1]\right)$$

`META` stock market data with bid-ask spreads

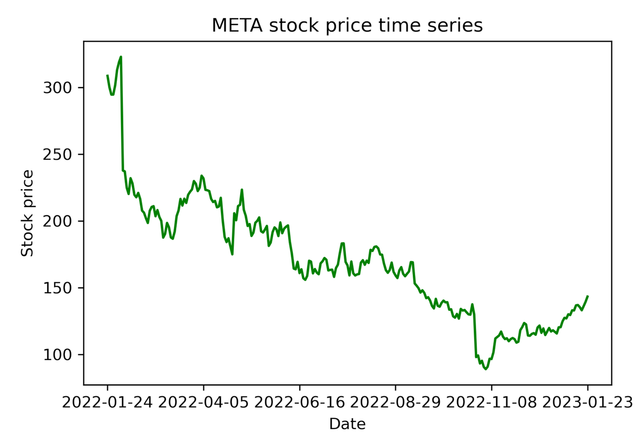
Consider a single risky asset:

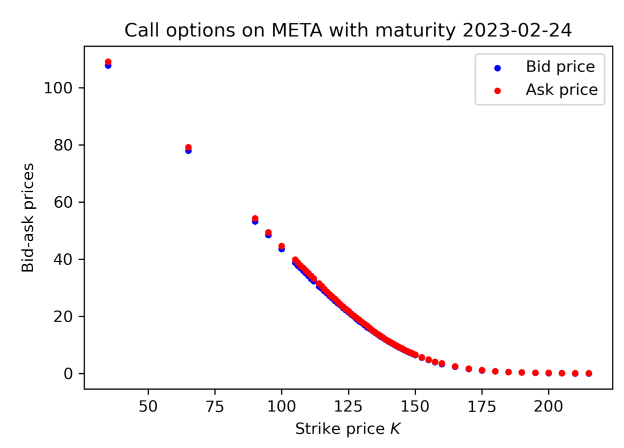
- $t = 0$ identified with $2023-01-23$
- $t = 1$ identified with $2023-02-24$
- US T-Bill with $1 + r = (1.0469)^{\frac{32}{365}}$
- Last one year of `META` closing prices:

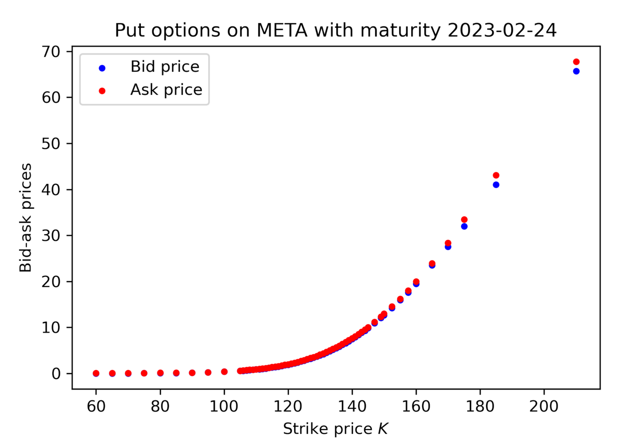
S_1^1 ranging in
$\mathcal{S}_1^1 = \{112.4, 159.2, 206.0, 252.8, 299.6\}$

- Bid-ask prices at time $t = 0$ of call and put options on `META` with maturity $t = 1$, strike prices in $\mathcal{K}_{call}$ and $\mathcal{K}_{put}$, and payoffs

$C_1^K = \max\{S_1^1 - K\}$
$P_1^K = \max\{K - S_1^1\}$







Tuning of α: a measure of market pessimism

For a fixed $\alpha \in [0,1]$, compute the α-mixture prices $C_0^{K,\alpha} = \alpha \underline{C}_0^K + (1-\alpha)\overline{C}_0^K$ and $P_0^{K,\alpha} = \alpha \underline{P}_0^K + (1-\alpha)\overline{P}_0^K$:

$$\text{minimize } E(\widehat{\varphi_\alpha}) = \sum_{K \in \mathcal{K}_{call}} \left(C_0^{K,\alpha} - \frac{\mathbb{C}_{\widehat{\varphi_\alpha}}[C_1^K]}{1+r}\right)^2 + \sum_{K \in \mathcal{K}_{put}} \left(P_0^{K,\alpha} - \frac{\mathbb{C}_{\widehat{\varphi_\alpha}}[P_1^K]}{1+r}\right)^2$$

$$\text{subject to: } \begin{cases} \widehat{\varphi_\alpha} \in \mathbf{M}_\alpha, \\ \widehat{\varphi_\alpha} \text{ is represented by } \widehat{Bel}, \\ \widehat{Bel}(\{i\}) \geq \epsilon, \text{ for all } i \in \Omega, \text{ with } \epsilon = 0.0001 \end{cases}$$