Indistinguishability through Exchangeability in Quantum Mechanics?

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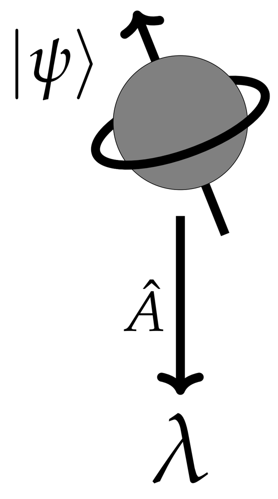


State $|\psi\rangle$: element of a finite dimensional Hilbert space $\mathscr{X}$.

Measurement $\hat{A}$: Hermitian operator on $\mathscr{X}$

Possible outcomes: the eigenvalues λ of $\hat{A}$

How can we model the uncertainty about $|\Psi\rangle$?

Density operators

- *Density operator* $\hat{\rho} := \sum_{k=1}^{r} p_k |\psi_k\rangle\langle\psi_k|$ with probability mass function $p_1, \dots, p_r$ over possible states $|\psi_1\rangle, \dots, |\psi_r\rangle$.
- *Expectation* of outcome of $\hat{A}$ is $\mathrm{E}_{\hat{\rho}}(\hat{A}) = \mathrm{Tr}(\hat{\rho}\hat{A})$.

Adding imprecision

Utility function

$$\hat{A} \longrightarrow u_{\hat{A}}(|\psi\rangle) = \langle\psi|\hat{A}|\psi\rangle$$

Possible motivations:
1) Decision theoretic postulates
2) Expected outcome of $\hat{A}$ given probabilities

Set of desirable measurements

A measurement $\hat{A}$ is desirable if You prefer the uncertain utility $u_{\hat{A}}(|\Psi\rangle)$ to receiving nothing.

D1. $\hat{0} \notin \mathscr{D}$; [strictness]
D2. $\mathscr{H}_{>\hat{0}} \subseteq \mathscr{D}$; [accepting sure gain]
D3. $\hat{A}, \hat{B} \in \mathscr{D} \Rightarrow \hat{A} + \hat{B} \in \mathscr{D}$; [additivity]
D4. $\hat{A} \in \mathscr{D} \Rightarrow \lambda\hat{A} \in \mathscr{D}$. [positive scaling]

Credal set

Set of density operators

$$\mathscr{R}_{\mathscr{D}} = \{\hat{\rho} \colon (\forall \hat{A} \in \mathscr{D})\, \mathrm{Tr}(\hat{\rho}\hat{A}) \geq 0\}.$$

Up to boundary behaviour

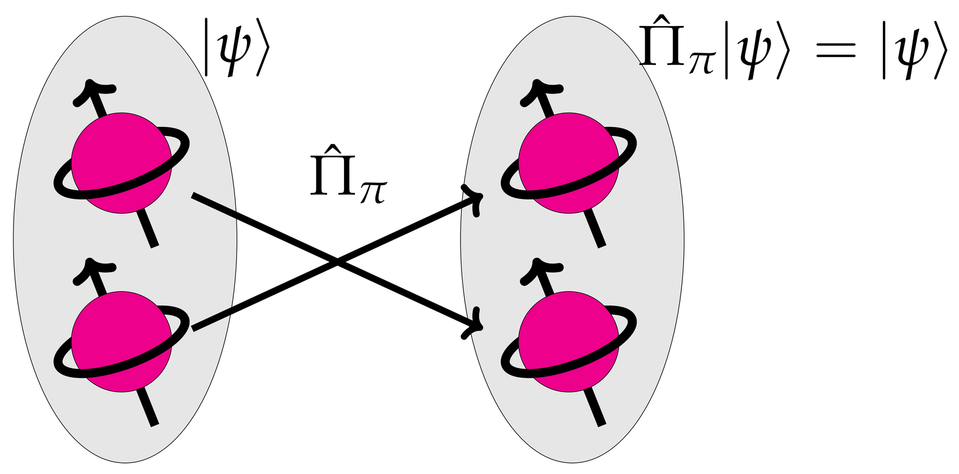
What changes when the particles in a system are identical?

Standard approach: **Symmetrisation postulate**

When a system is made up of several identical particles, only certain kets in its state space can describe its physical states. Physical states are, depending on the nature of the identical particles, either symmetric or antisymmetric with respect to permutation of these particles. Those particles for which the physical states are symmetric are called **bosons**, and those for which they're antisymmetric, **fermions**.

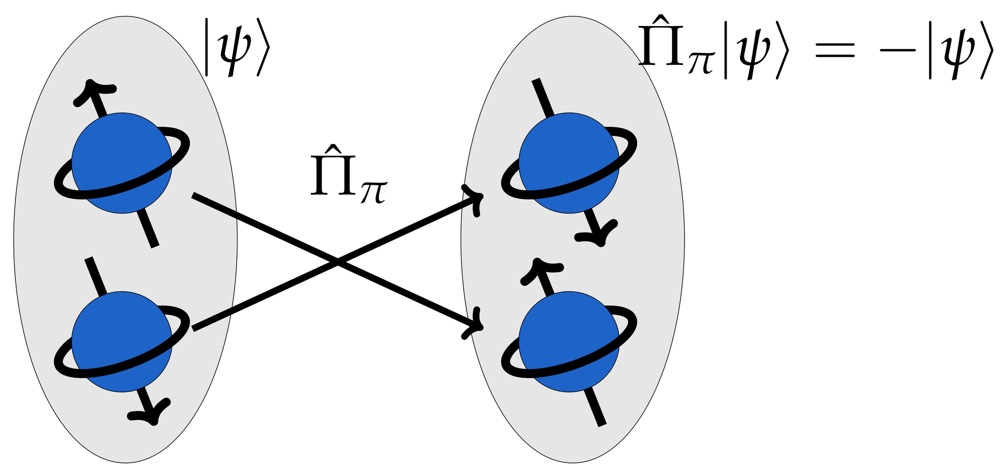
Bosons (Symmetric particles)

State $|\psi\rangle \in \mathscr{X}_{\mathrm{s}}$: $\hat{\Pi}_\pi|\psi\rangle = |\psi\rangle$ for all π.
Bosonic density operator $\hat{\rho}_{\mathrm{s}}$: $\hat{\rho} = \hat{\Pi}_\pi\hat{\rho} = \hat{\rho}\hat{\Pi}_\pi^\dagger$ for all π.



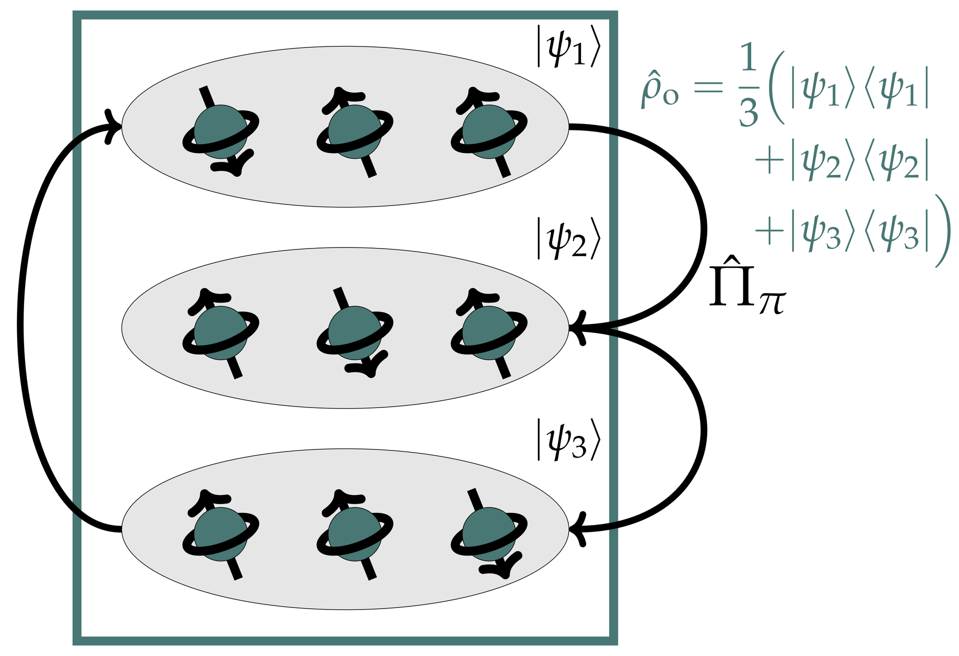
Fermions (Antisymmetric particles)

State $|\psi\rangle \in \mathscr{X}_{\mathrm{a}}$: $\hat{\Pi}_\pi|\psi\rangle = \mathrm{sgn}(\pi)|\psi\rangle\}$ for all π.
Fermionic density operator $\hat{\rho}_{\mathrm{a}}$: $\hat{\rho} = \mathrm{sgn}\,\pi\hat{\Pi}_\pi\hat{\rho} = \mathrm{sgn}\,\pi\hat{\rho}\hat{\Pi}_\pi^\dagger$ for all π.



Para particles (HUH?)

These obey a weaker permutation symmetry. As a collection of paraparticles cannot be distuinguished from standard particles, there existence has neither been confirmed nor excluded.
In fact, some look to parastatistics in order to explain some properties of dark matter.



Idea: indistinguishability through indifference

Exchangeability
Identical particles $\Longrightarrow$ order should not matter.
Indifferent between receiving $u_{\hat{A}}(|\Psi\rangle)$ and $u_{\hat{A}}(\hat{\Pi}_\pi|\Psi\rangle)$, or equivalently between executing $\hat{A} - \hat{\Pi}_\pi^\dagger\hat{A}\hat{\Pi}_\pi$ and $\hat{0}$.

Strong $\star$-symmetry
How can we single out the assessment that the particles are bosons or fermions? We need to break the quadratic symmetry.

$$S_\pi^\star(\hat{A}) := \frac{\mathrm{sgn}^\star(\pi)}{2}(\hat{\Pi}_\pi^\dagger\hat{A} + \hat{A}\hat{\Pi}_\pi), \text{ for } \pi \in \mathbb{P}.$$

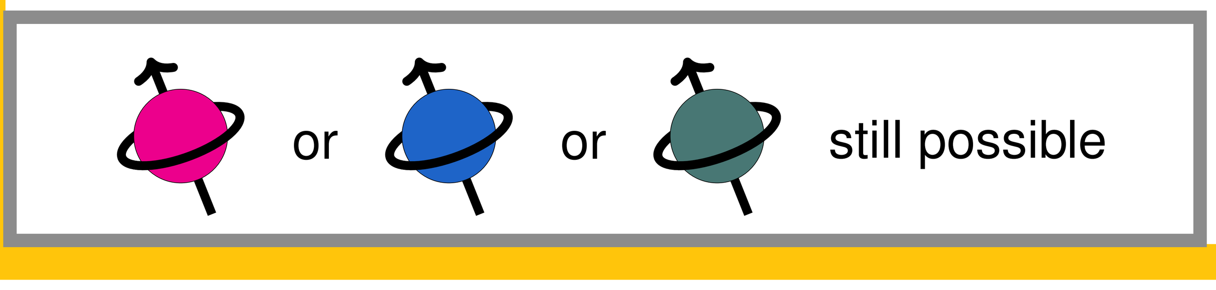
$$\mathscr{I} = \{\hat{A} - \hat{\Pi}_\pi^\dagger\hat{A}\hat{\Pi}_\pi \colon \hat{A} \in \mathscr{H}, \pi \in \mathbb{P}\}$$

Indifference
$$\mathscr{D} + \mathscr{I} \subseteq \mathscr{D}$$

$$\mathscr{I}^\star := \{\hat{A} - S_\pi^\star(\hat{A}) \colon \hat{A} \in \mathscr{H}, \pi \in \mathbb{P}\}$$

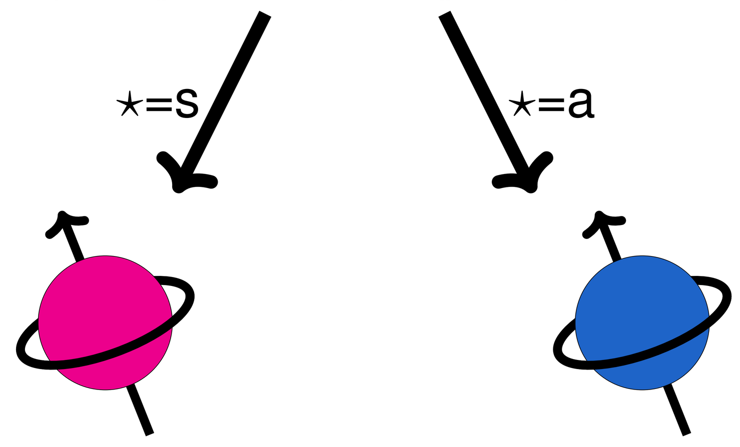
Credal set
Credal set is exchangeable if every density operator is permutation invariant: $\hat{\rho} = \hat{\Pi}_\pi\hat{\rho}\hat{\Pi}_\pi$ for all $\pi \in \mathbb{P}$.
Equivalently, these density operators can be written as the the probabilistic mixing of a bosonic, a fermionic and a paraparticle density operator

$$\hat{\rho} = p_s\hat{\rho}_{\mathrm{s}} + p_a\hat{\rho}_{\mathrm{a}} + p_o\hat{\rho}_{\mathrm{O}}.$$



Credal set
A credal set is strongly $\star$-symmetric if for every density operator

$$\hat{\rho} = \mathrm{sgn}^\star(\pi)\hat{\Pi}_\pi\hat{\rho} = \mathrm{sgn}^\star(\pi)\hat{\rho}\hat{\Pi}_\pi.$$



Second quantisation

There is a lot of redundancy in the description of fermions and bosons.
In quantum mechanics, this is solved by considering second quantisation. This approach considers occupation numbers and is completely similar to the count vector approach in imprecise probabilities. In fact, we proved that we can represent strong $\star$-symmetric sets of desirable measurements on the whole $\mathscr{X}$, by smaller dimensional sets of desirable measurements on $\mathscr{X}_\star$.

¿¿¿FLip quiz???

What kind of particle is not modelled through strong $\star$-symmetry?