A Pointfree Approach to Measurability and Statistical Models

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The problem

In this work we approach the problem of finding the most natural algebraic structure of the set of all possible random variables on a measurable space, inspired by Nelson's and Rota's point of view.

Building on a recent paper by D. Mundici, in which he argues that MV-algebras provide the natural algebraic setting for continuous random variables, we tackle the issue of dealing with non-necessarily continuous RVs.

Algebraic framework

The main character is a variety of algebras called σ-continuous Riesz MV-algebras. These form an infinitary variety called $\mathbf{RMV}_\sigma$ and here it is enough think of them as algebras of continuous functions that are, additionally, closed under countable suprema. Indeed, $A \in \mathbf{RMV}_\sigma$ iff $A = C(X)$, where X is a basically disconnected compact Hausdorff space and functions in $C(X)$ are continuous are $[0,1]$-valued.

Free algebras in $\mathbf{RMV}_\sigma$ have been characterized as algebras of measurable functions. If $\kappa \leq \omega$, then the free κ-generated algebra in $\mathbf{RMV}_\sigma$ is the algebra of all Borel measurable functions $\mathsf{Borel}([0,1]^\kappa)$.

Furthermore, we denote by H, I, S and P the standard universal-algebraic operators in $\mathbf{RMV}_\sigma$, that is, taking homomorphic images, taking isomorphic images, taking subalgebras and taking products. Then, it follows that $\mathbf{RMV}_\sigma = HSP([0,1])$. We call σ-semisimple any algebra in $ISP([0,1])$.

Main results

1. Algebras of measurable functions, topologically and algebraically

If $\mathcal{B}$ is any σ-algebra on a set B, let us denote by $\mathsf{Meas}(B,[0,1])$ the σ-complete Riesz MV-algebra of measurable functions from B to $[0,1]$, where we take the unit interval endowed with the σ-algebra of its Borel subsets.

Theorem 1. *A σ-complete Riesz MV-algebra A is isomorphic to a Riesz MV-algebra of the form $\mathsf{Meas}(B,[0,1])$ if and only if A is σ-semisimple.*

If (X,τ) ia a topological space, we denote by $\overline{S}$ the closure of a set S and we denote by $\mathsf{coz}\,(f)$ the cozero of a function $f\colon X \to [0,1]$, that is, the set of points of X where f is nonzero. We denote by $\mathsf{coz}\,(X)$ the set of all subsets of X that are the cozero of some continuous function.

Proposition 2. *The algebra $A := C(X)$ is σ-semisimple if and only if $\bigcap_{C\in\mathsf{coz}(X)}(\mathsf{int}(X\setminus C)\cup C)$ is dense, where $\mathsf{int}(X\setminus C)$ denotes the interior of $X\setminus C$.*

2. A pointfree approach

Traditionally, a topology is defined starting from a set X and the collection $\mathcal{O}$ of its open subsets. After Stone's famous duality result between certain topological spaces and boolean algebras, it became more clear that one could discuss topology taking as primitive the notion of open set, rather than the one of point. With this idea in mind, topology was reconsidered pointfree, using the notion of a frame, that is, a bounded complete lattice in which finite meets distribute over arbitrary joins. Via dualities, compact Hausdorff spaces correspond to the so-called compact and regular frames. For brevity, $\Omega(I)$ denotes the frame of open subsets of $[0,1]$ with the Euclidean topology.

Theorem 3. *For any σ-complete Riesz MV-algebra A there exists a frame L such that $A \simeq \mathsf{Hom}_{\mathsf{KRF}}(\Omega(I),L)$, the set of frame homomorphisms between the frames $\Omega(I)$ and L.*

For any frame L, a *cozero* is an element $c \in L$ such that there exists a frame homomorphism $h : \Omega(I) \to L$ and $c = h((0,1])$. Denote the set of all cozeros by $\mathsf{coz}\,(L)$.

Corollary 4. *An algebra $A \in \mathbf{RMV}_\sigma$ is σ-semisimple if and only if the associate frame L (wrt Theorem 3) is compact, regular and it satisfies*

$$\left(\bigwedge_{c\in\mathsf{coz}(L)}(c^*\vee c)\right)^{**} = \top,$$

where, for any $x\in L$, $x^* := \bigvee\{y \in L \mid x\wedge y = \bot\}$.

Logico-algebraic statistical models

Formally, for a $\kappa \leq \omega$ a statistical model is a function $\eta = (\eta_i)_{i\in\kappa}\colon P \to \Delta_\kappa$, where $P \subseteq [0,1]^d$ is an intersection of Borel measurable sets and Δ_κ is the standard κ-dimensional simplex. When $\kappa = \omega$, we take Δ_ω to be $\{x \in [0,1]^\omega \mid \sum_{i=1}^\infty x_i \leq 1\}$. We call κ-dimensional any statistical model whose codomain is Δ_κ. The intuition behind this definition is the following:

- $[0,1]^\kappa$ is the set of observations on the real world and $\mathsf{Borel}([0,1]^\kappa)$ is the algebra of many-valued events, while the set $P \subseteq [0,1]^d$ is the set of states of the world, or parameters, we allow d to be any countable cardinal;
- the tuple of functions $\eta := (\eta_i)_{i\in\kappa}\colon P \to [0,1]^\kappa$ is our statistical model: to each parameter $\mathbf{x} \in P$ it associates the tuple $(\eta_i(\mathbf{x}))_{i\in\kappa}$. Each $\eta_i\colon [0,1]^d \to [0,1]$ is a Borel measurable function.

Proposition 5. *Let A be a σ-semisimple algebra with at most a countable number of generators and $\kappa \leq \omega$. There exists a set of parameters P_A such that the set of homomorphisms between $\mathsf{Borel}(\Delta_\kappa)$ and A is isomorphic to the set of all κ-dimensional statistical models defined on P_A.*

Category Theory and Future Work

It is possible to define a functor that, to each A assigns the set of all κ-dimensional statistical models, for a fixed κ. This functor is actually a presheaf and the last result of the paper, that is skipped in this poster, gives a categorical interpretation of statistical models that paves the way for a study on this notion from the perspective of sheaves.

Moreover, when $A = \mathsf{Borel}(\Delta_\kappa)$, each probability on Δ_κ can be seen as a function that maps a μ-dimensional statistical model on A to the set of all probabilities on Δ_μ, which is turn can be seen as the set of all probabilities on μ points.