A New Outer Measure for Model-Free Generalized Processes

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Applications in Mathematical Finance

- The goal: to describe all possible paths representing evolution of prices of traded financial assets which do not lead to **arbitrage**.
- The 'old' tool: an outer measure on **sets (properties) of continuous paths** $\omega : [0,T] \to \mathbb{R}$. This captures the following notion of arbitrage: **no infinite gains at the terminal time T by risking \$1 only**.
- The 'new' tool: an outer measure on **sets (properties) of pairs** (t, ω), where $t \in [0, +\infty)$ represents **time** and $\omega \in \Omega$ represents **elementary event** (the 'state of the world' or the outcome of reality). This captures the following notion of arbitrage: **no infinite gains achieved by risking only \$1 immedietely after a given property ceases to hold**.

Background

The investigations in game-theoretic approach to model-free, financial models of continuous price paths culminated in Glenn Shafer and Vladimir Vovk publishing their book [SV19]. In their book Shafer and Vovk introduce a new notion – the notion of *instant enforcement*. But they do not characterize it using any outer measure, unlike what Vovk did in the case of sets of "typical" (not leading to arbitrage) price paths. Informally, property E is instantly enforceable if there exists a trading strategy making a trader using this strategy infinitely rich as soon as the property E ceases to hold. In [Łoc22] I introduce an outer measure on the space $[0, +\infty) \times \Omega$, which assigns zero value exactly to those sets (properties) of pairs of time t and an elementary event ω, complements of which are instantly enforceable.

Model-free martingale spaces

We will work with a martingale space which is a quintuple

$$\left(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, J = \{1, 2, \ldots, d\}, \left\{S^j, j \in J\right\}\right)$$

of the following objects:

- Ω is a space of possible outcomes of reality, whose elements are called *elementary events*,
- $\mathcal{F}$ is a σ-field of the subsets of Ω which we call *events*,
- $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is a *filtration* (this represents the influx of information in time) such that for $t \geq 0$, $\mathcal{F}_t \subseteq \mathcal{F}$,
- and $\left\{S^j, j \in J\right\} = \left\{S^1, S^2, \ldots S^d\right\}$ is a family of mappings $S^j : [0, +\infty) \times \Omega \to \mathbb{R}$, called *basic continuous martingales*, such that for any $t \geq 0$ and $j \in J$, S^j_t is a $(\mathcal{F}_t, \mathcal{B}(\mathbb{R}))$-measurable *real variable* $S^j_t : \Omega \to \mathbb{R}$ ($\mathcal{B}(\mathbb{R})$ denotes the σ-field of Borel subsets of the set of real numbers $\mathbb{R}$) and such that for each $\omega \in \Omega$ the trajectory $[0, +\infty) \ni t \mapsto S^j_t(\omega)$ is continuous.

Stopping times

An $\mathbb{F}$-*stopping time* (or *stopping time* in short) is a random variable $\tau : \Omega \to [0, +\infty]$ such that for any $t \geq 0$,

$$\{\tau \leq t\} := \{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}_t.$$

A σ-field $\mathcal{F}_\tau$ generated by the stopping time τ consists of those events $A \in \mathcal{F}$ which for any $t \geq 0$ satisfy

$$A \cap \{\tau \leq t\} \in \mathcal{F}_t.$$

Trading strategies

A *simple trading strategy* is a triplet $(c, (\tau_n), (g_n))$ which consists of

- the **initial capital** $c \in \mathbb{R}$,
- a **sequence of $\mathbb{F}$-stopping** times (τ_n) such that
$$0 \equiv \tau_0 \leq \tau_1 \leq \tau_2 \ldots$$
- and a **sequence of globally bounded, real variables** $g_n : \Omega \to \mathbb{R}$, $n = 0, 1, \ldots$, such that g_n is $(\mathcal{F}_{\tau_n}, \mathcal{B}(\mathbb{R}))$-measurable and $g_n(\omega) = 0$ whenever $\tau_n(\omega) = +\infty$.

Capital processes

For a real process $X : [0, +\infty) \times \Omega \to \mathbb{R}$ and a simple trading strategy $G = (c, (\tau_n), (g_n))$ we define

$$(G \cdot X)_t(\omega) := c + \sum_{n=1}^{+\infty} g_{n-1}(\omega) \left(X_{\tau_n(\omega)\wedge t}(\omega) - X_{\tau_{n-1}(\omega)\wedge t}(\omega)\right).$$

(For two numbers $a, b \in [-\infty, +\infty]$ we define $a \wedge b = \min\{a, b\}$.)
We define the *simple capital process or simple integral* corresponding to the vector $\mathbf{G} = \left(G^j\right)_{j\in J}$ of simple trading strategies G^j, $j \in J$, as

$$(\mathbf{G} \cdot \mathbf{S})_t(\omega) := \sum_{j\in J} (G^j \cdot S^j)_t(\omega).$$

The simple capital process has a very natural interpretation – it is the wealth accumulated till time t by the application of the simple trading strategy G^j to the asset whose price is represented by the basic martingale S^j, $j \in J$.

Nonnegative supermartingales and instantly enforceable properties

The class $\mathcal{C}$ of *nonnegative supermartingales* is defined as the smallest class with the following properties

1. $\mathcal{C}$ contains all simple capital processes which are nonnegative;
2. whenever $X \in \mathcal{C}$, Y is a simple capital process and $X+Y$ is nonnegative then $X+Y \in \mathcal{C}$;
3. for any sequence (X^n) such that $X^n \in \mathcal{C}$ for $n \in \mathbb{N}$, we have that $X := \liminf_{n\to+\infty} X^n$ also belongs to $\mathcal{C}$.

A property $E \subseteq [0, +\infty) \times \Omega$ is *instantly enforceable*, or holds *with instant enforcement*, w.i.e. in short, if there exists $X \in \mathcal{C}$ such that $X_0 = 1$ and

$$(t, \omega) \notin E \implies X_t(\omega) = +\infty.$$

Complements of instantly enforceable properties (sets) are called *instantly blockable*.

A new outer measure and outer expectation

For $A \subseteq [0, +\infty) \times \Omega$ and for $Y : [0, +\infty) \times \Omega \to [-\infty, +\infty]$ we define

$$\overline{\mathbb{P}}(A) := \inf\left\{X_0 : X \in \mathcal{C} \text{ and } \forall (t, \omega) \in [0, +\infty) \times \Omega, X_t(\omega) \geq \mathbf{1}_A(t, \omega)\right\},$$

$$\overline{\mathbb{E}}Y := \inf\left\{X_0 : X \in \mathcal{C} \text{ and } \forall (t, \omega) \in [0, +\infty) \times \Omega, X_t(\omega) \geq Y_t(\omega)\right\}.$$

Lemma 0.1 *The set $B \subseteq [0, +\infty) \times \Omega$ is instantly blockable iff*

$$\overline{\mathbb{P}}(B) = 0.$$

References

[Łoc22] R. Łochowski. BDG inequalities for model-free continuous price paths with instant enforcement. *Preprint arXiv:2109.07928*, 2022.

[SV19] Glenn Shafer and Vladimir Volk. *Game-Theoretic Foundations for Probability and Finance.* Wiley Series in Probability and Statistics. Wiley, 2019.