Neighbourhood models induced by the Euclidean distance and the Kullback-Leibler divergence

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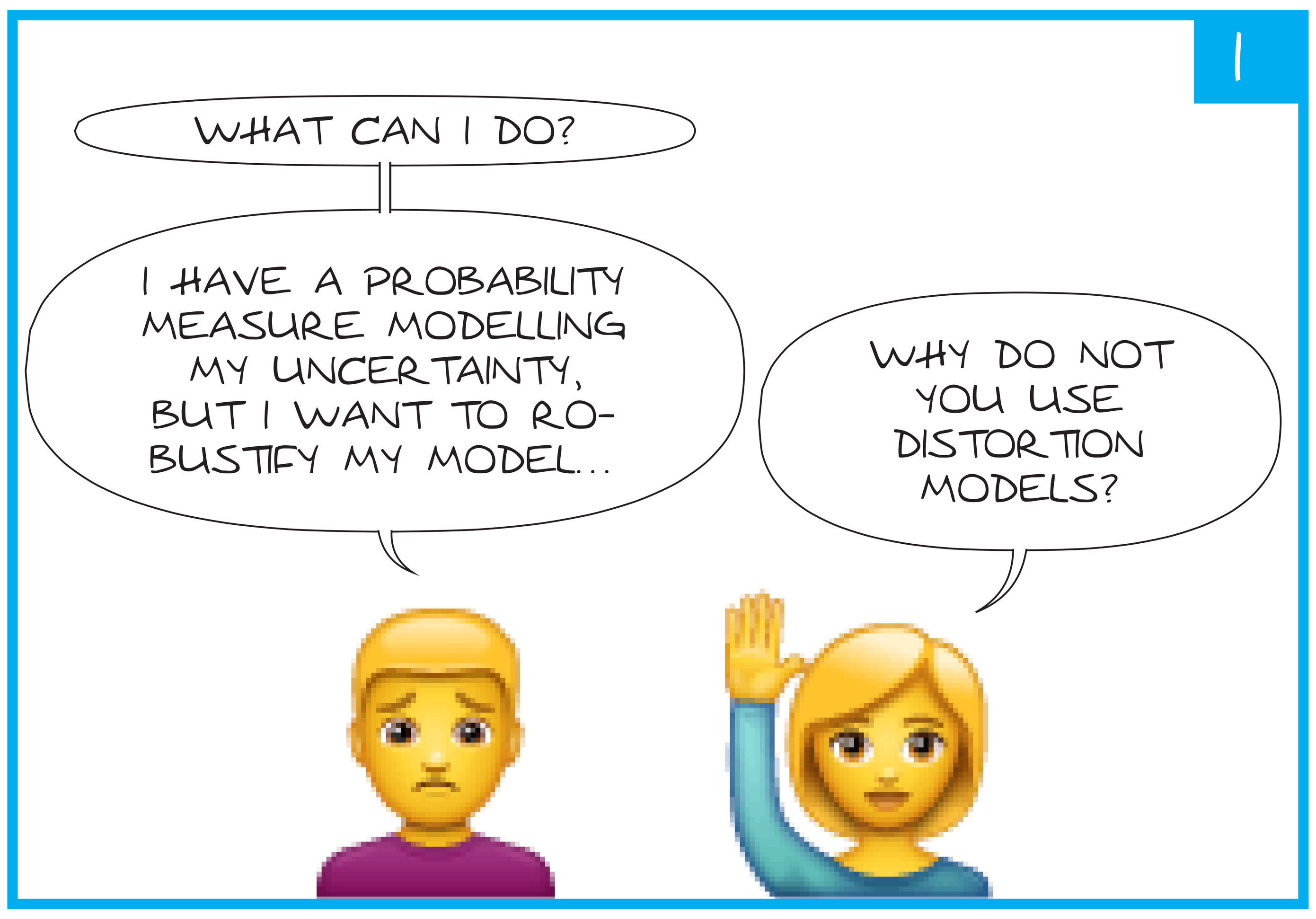


DISTORTION MODELS

1. Ingredients

P_0: probability measure

δ: distortion parameter

d: distorting function

Assumptions:

$\mathcal{X} = \{x_1, \ldots, x_n\}$

$P_0(A) > 0 \ \forall A \neq \emptyset$

δ *"small enough"*

2. The model

$$B_d^\delta(P_0) = \{P \mid d(P, P_0) \leq \delta\}$$

$\downarrow$

$$\underline{P}_d(f) = \inf\{P(f) \mid P \in B_d^\delta(P_0)\}$$

$\downarrow$

$$\mathcal{M}(\underline{P}_d) = \{P \mid P(f) \geq \underline{P}_d(f) \ \forall f\}$$

d continuous and convex

3. Particular cases

3.1 Known IP-models

$\underline{P}_{PMM}(A) = \max\{0, (1+\delta)P_0(A) - \delta\}$

$\underline{P}_{LV}(A) = (1-\delta)P_0(A)$ $\quad A \subset \mathcal{X}$

$\underline{P}_{COR}(A) = \frac{(1-\delta)P_0(A)}{1-\delta P_0(A)}$

3.2 Distance-based models

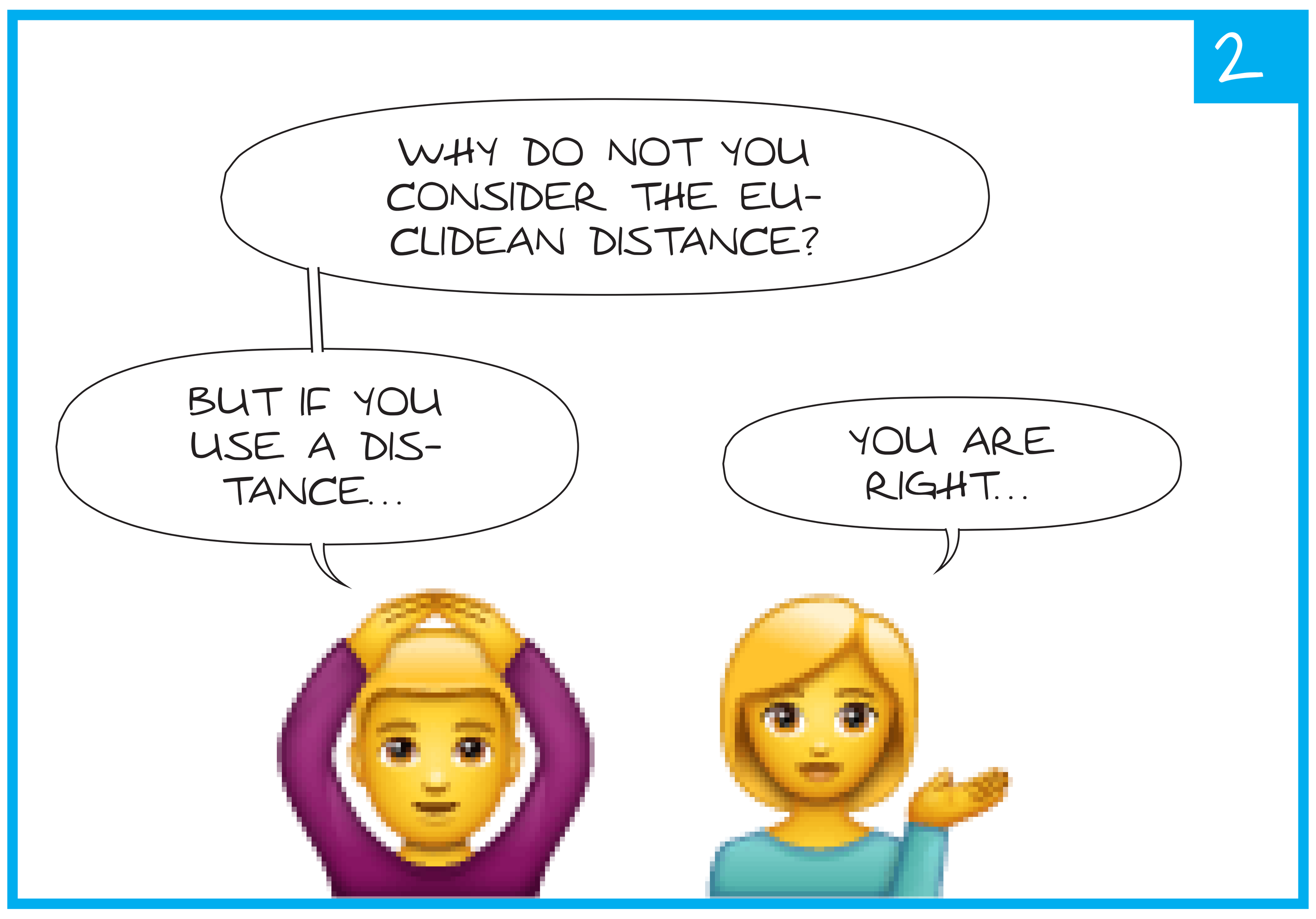
$\underline{P}_{TV}(A) = \max\{0, P_0(A) - \delta\}$ $\quad A \subset \mathcal{X}$

3.3 Vertical barrier models

$\underline{P}_{VB}(A) = \max\{bP_0(A) + a, 0\}$ $\quad a \leq 0,\ b > 0,\ a + b \in [0,1]$

3.4 Increasing transformations

$\underline{P}(A) = g(P_0(A))$ $\quad g: [0,1] \to [0,1]$ increasing, $g(0) = 0,\ g(1) = 1$



EUCLIDEAN MODEL

1. Euclidean distance

$$d_E(P, P_0) = \sqrt{\sum_{i=1}^n (p_i - p_i^0)^2}$$

d_E continuous and convex

2. Lower prevision

$$f = \sum_{i=1}^n a_i I_{x_i} \quad \begin{cases} \overline{f} = \frac{1}{n}\sum_{i=1}^n a_i \\ S_f^2 = \frac{1}{n}\sum_{i=1}^n (a_i - \overline{f})^2 \end{cases}$$

$$\underline{P}_E(f) = P_0(f) - \delta\sqrt{n}S_f$$

3. Lower probability

$$\underline{P}_E(A) = P_0(A) - \delta\sqrt{\frac{|A|(n-|A|)}{n}} \quad A \subset \mathcal{X}$$

4. Properties

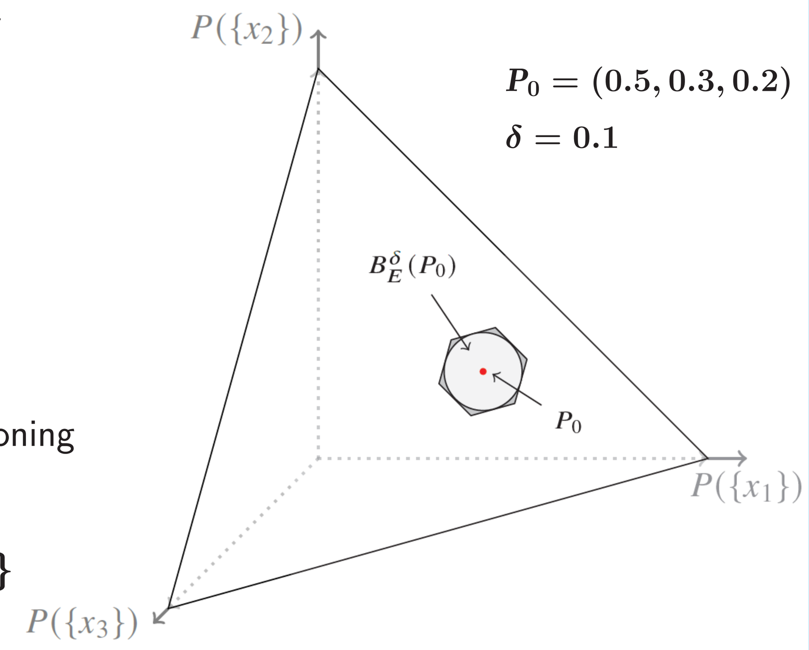
1) $\underline{P}_E$ is not 2-monotone...
2) ...but it is 2-monotone on events
3) the model is not preserved under conditioning
4) extreme points:

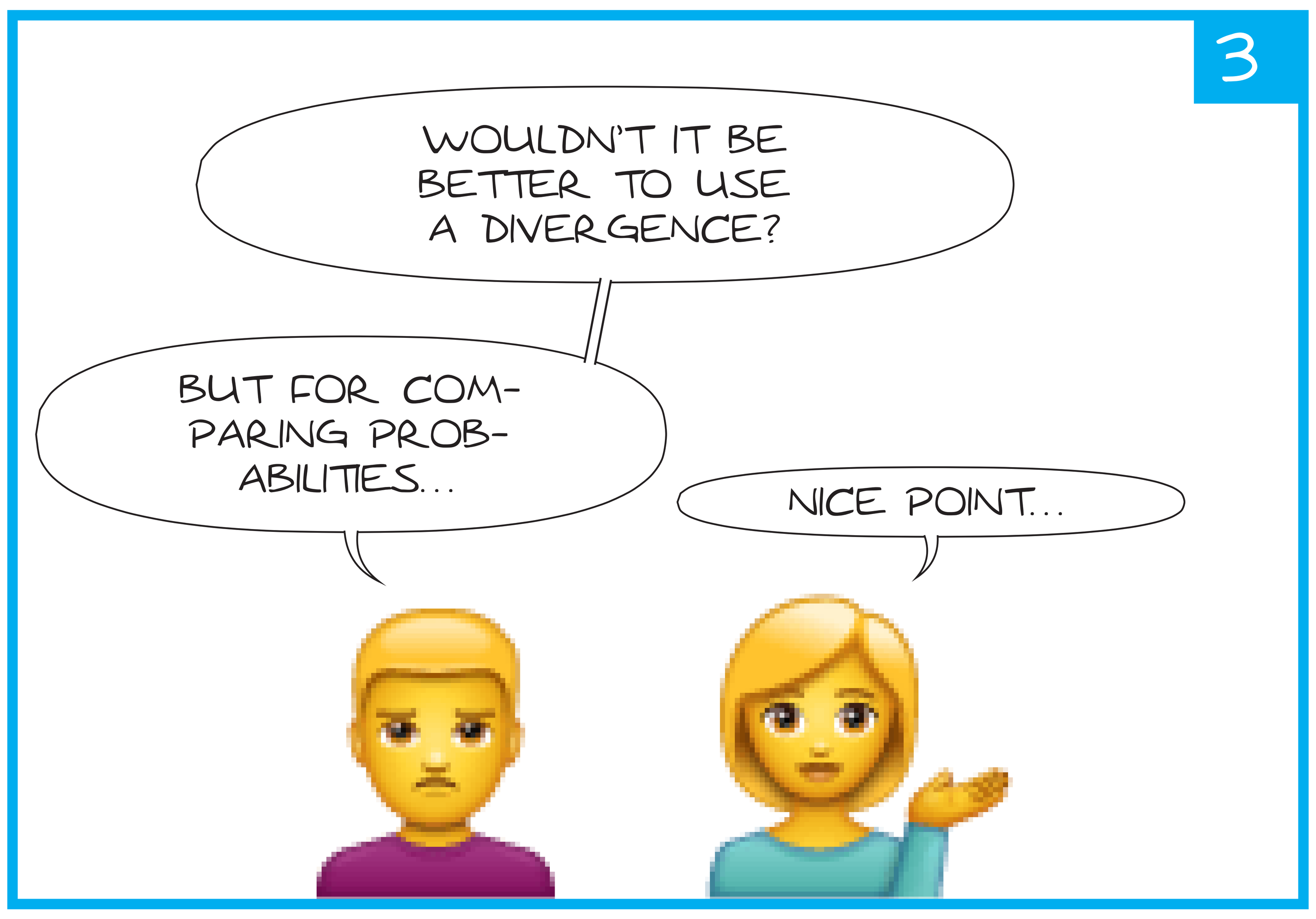
$$\text{ext}(B_E^\delta(P_0)) = \{P \mid d_E(P, P_0) = \delta\}$$

5. Example

$P_0 = (0.5, 0.3, 0.2)$

$\delta = 0.1$





KULLBACK-LEIBLER MODEL

1. KL-divergence

$$D_{KL}(P, P_0) = \sum_{i=1}^n p_i \log\left(\frac{p_i}{p_i^0}\right)$$

D_{KL} continuous and convex

2. Lower prevision

$$f = \sum_{i=1}^n a_i I_{x_i}$$

$$\underline{P}_{KL}(f) = \min_{\alpha_1,\ldots,\alpha_n} \sum_{i=1}^n a_i \alpha_i p_i^0$$

subject to

$$\sum_{i=1}^n \alpha_i p_i^0 = 1, \quad \sum_{i=1}^n \alpha_i p_i^0 \log(\alpha_i) = \delta$$

3. Lower probability

$\underline{P}_{KL}(A) = \alpha P_0(A)$

where $\alpha \in [0,1]$ is the unique solution of

$$\alpha P_0(A)\log(\alpha) + (1 - \alpha P_0(A))\log\left(\frac{1-\alpha P_0(A)}{1-P_0(A)}\right) = \delta$$

4. Properties

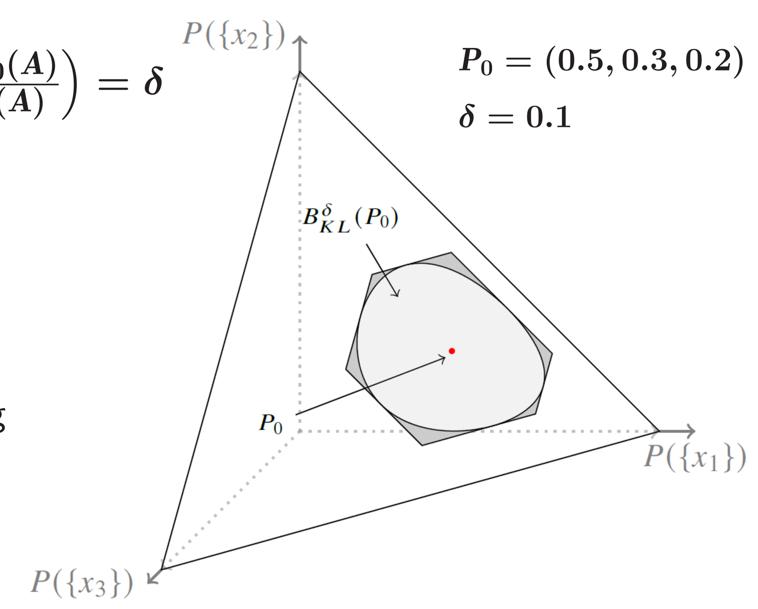
1) $\underline{P}_{KL}$ is not 2-monotone...
2) ...but it is 2-monotone on events
3) the model is not preserved under conditioning
4) extreme points:

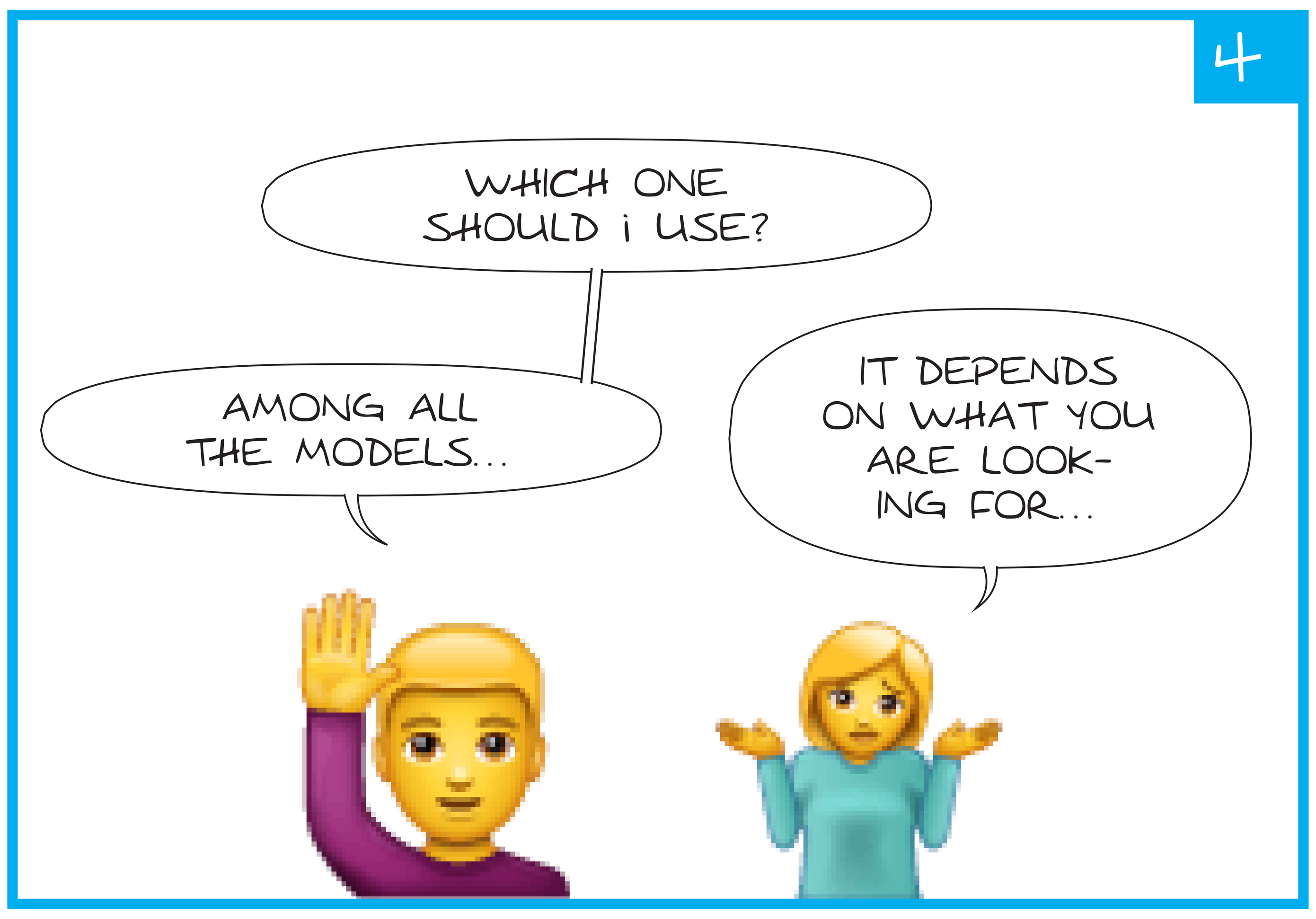
$$\text{ext}(B_{KL}^\delta(P_0)) = \{P \mid D_{KL}(P, P_0) = \delta\}$$

5. Example

$P_0 = (0.5, 0.3, 0.2)$

$\delta = 0.1$



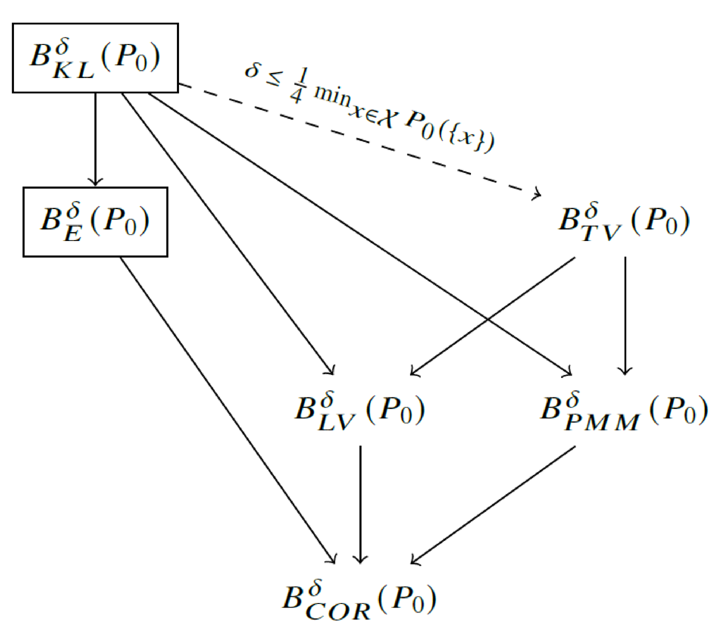


COMPARISON

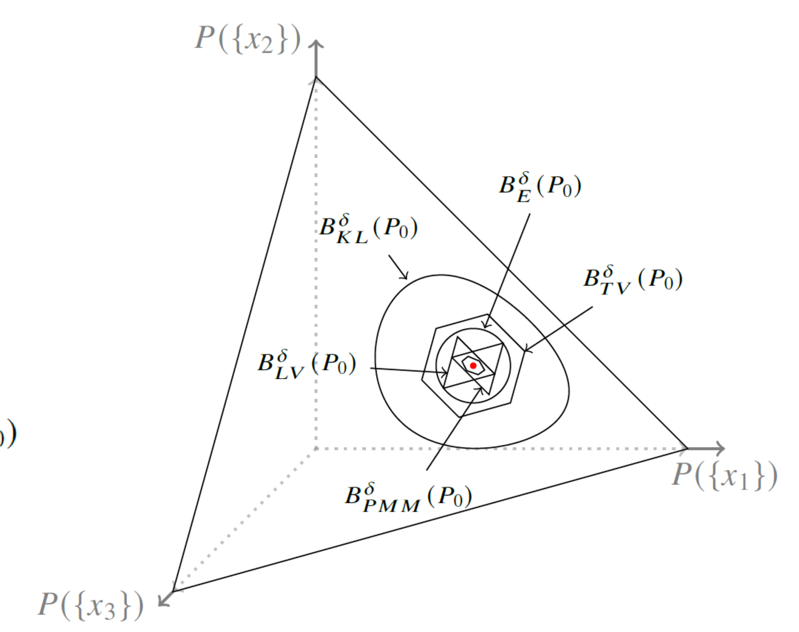
1. Properties

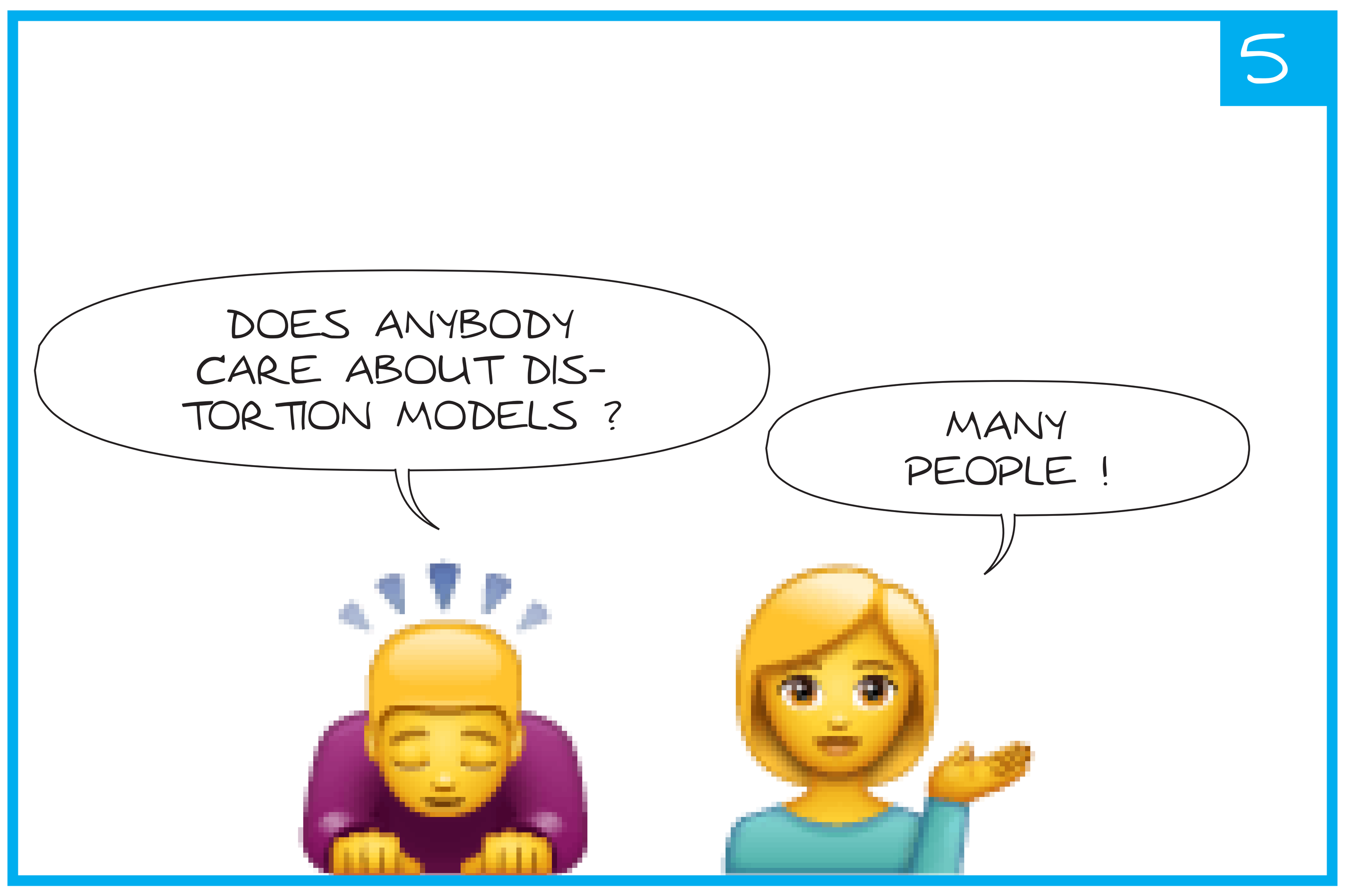
	2-monot.	∞-monot.	Prob.interval	Conditioning
Euclidean-model...				
...on gambles	✗	✗	✗	✗
...on events	✓	✗	✗	✗
KL-model...				
...on gambles	✗	✗	✗	✗
...on events	✓	✗	✗	✗
PMM	✓	✗	✓	✓
LV	✓	✓	✓	✓
COR...				
...on gambles	✗	✗	✗	✓
...on events	✓	✓	✗	✓
TV	✓	✗	✓	✓
Vertical Barrier models	✓	✗	✗	✓

2. Amount of imprecision



3. Example





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