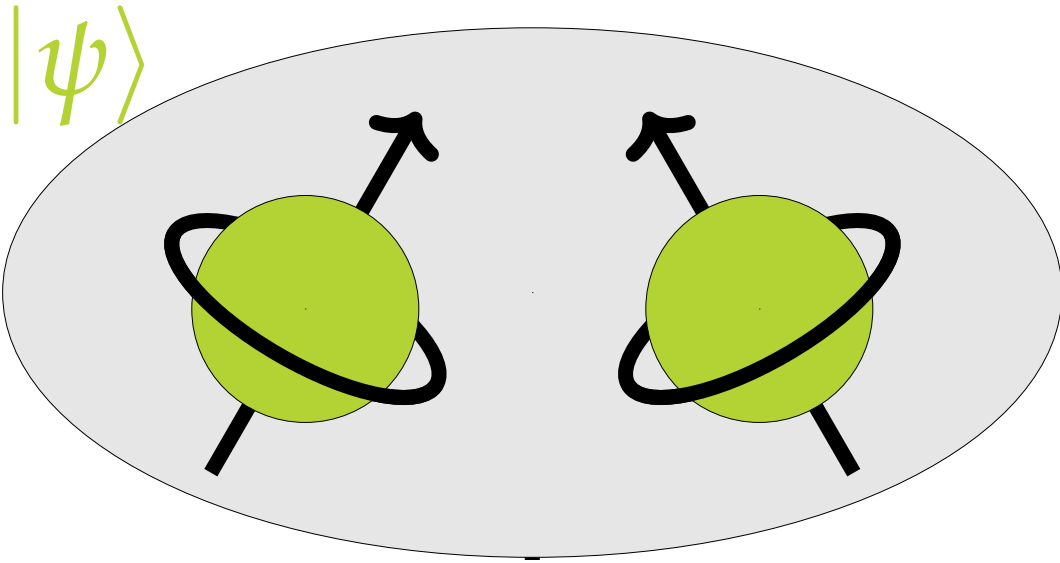


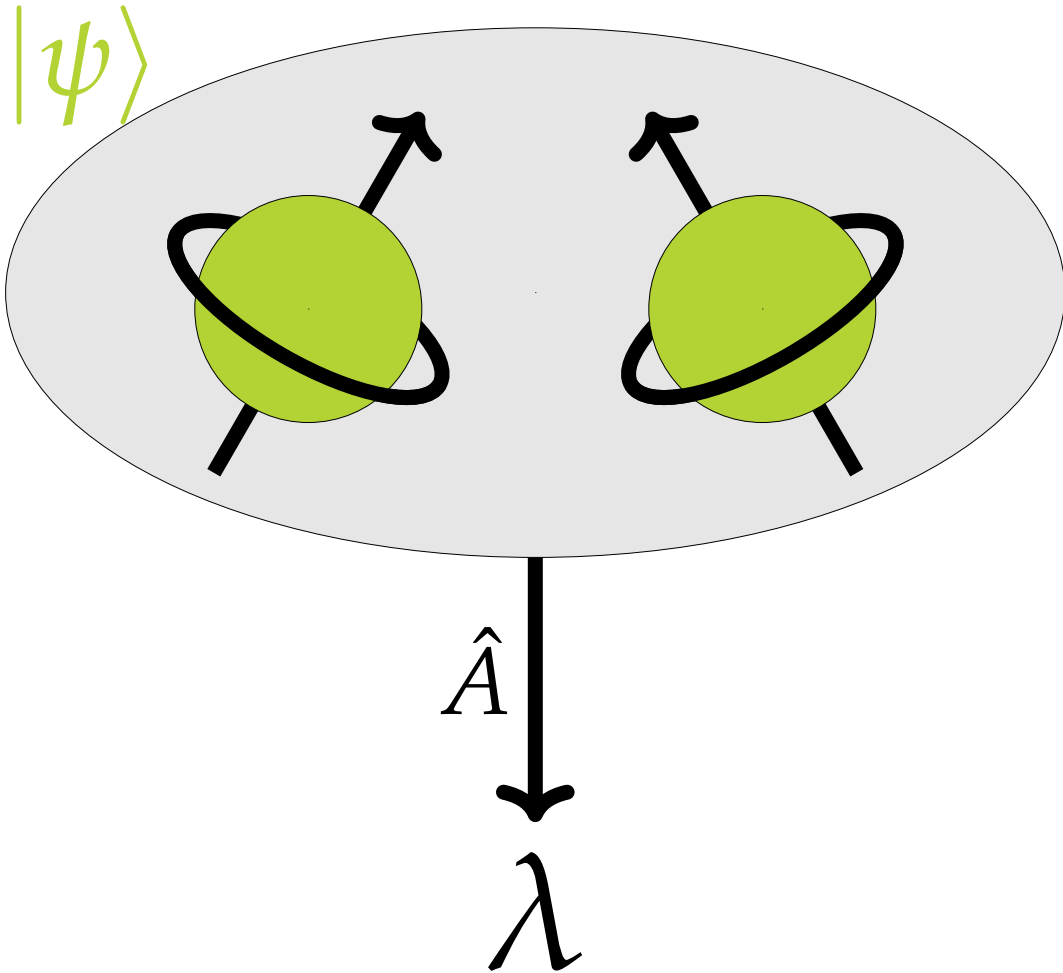
Indistinguishability through  
Exchangeability in Quantum  
Mechanics?

# Quantum Mechanics



*State*  $|\psi\rangle$ :  
element of a finite dimensional  
Hilbert space  $\mathcal{X}$

# Quantum Mechanics



*State*  $|\psi\rangle$ :  
element of a finite dimensional  
Hilbert space  $\mathcal{X}$

*Measurement*  $\hat{A}$ :  
Hermitian operator on  $\mathcal{X}$

*Possible outcomes*:  
the eigenvalues  $\lambda$  of  $\hat{A}$

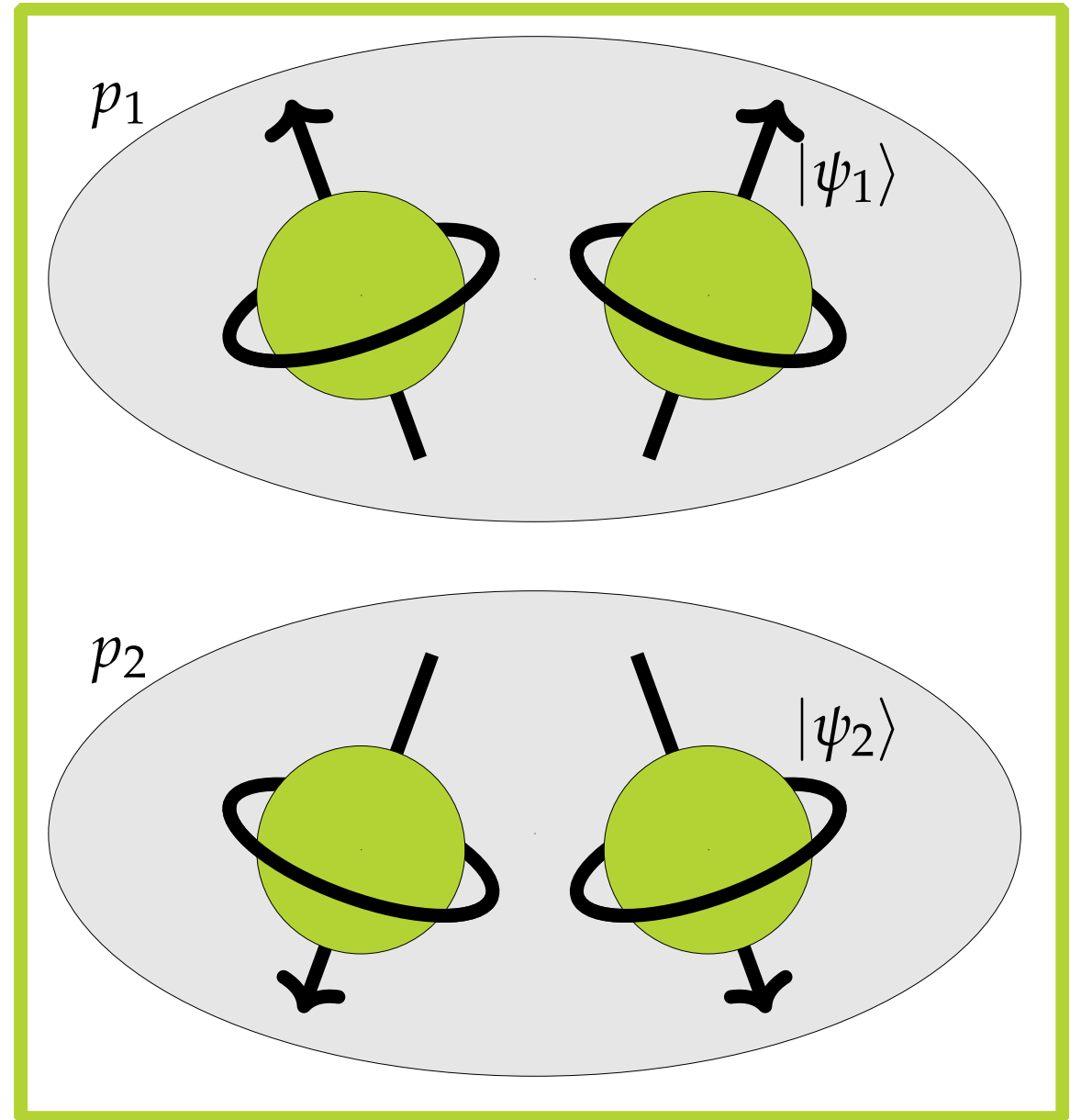
# Uncertainty about the state

Density operator  $\hat{\rho} := \sum_{k=1}^r p_k |\psi_k\rangle \langle \psi_k|$

With

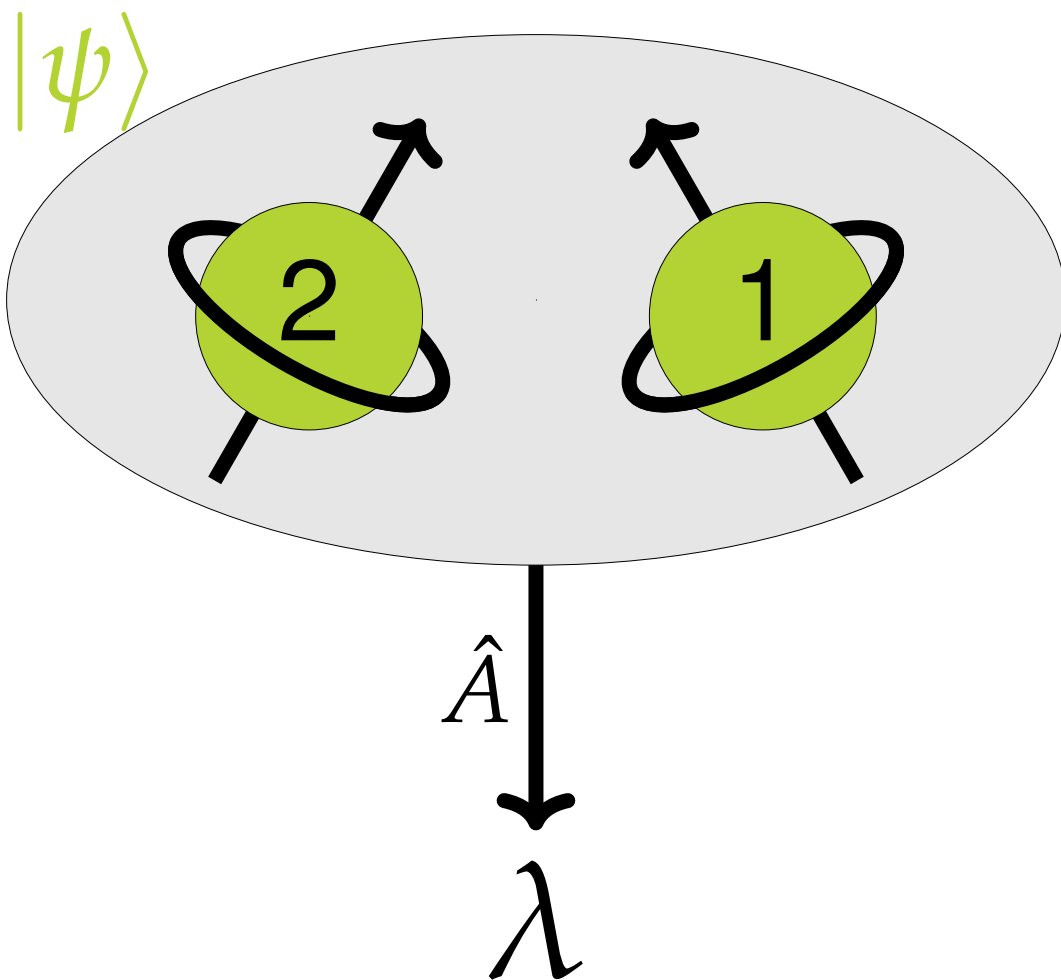
Probabilities  $p_1, \dots, p_r$

States  $|\psi_1\rangle, \dots, |\psi_r\rangle$



$$\hat{\rho} = p_1 |\psi_1\rangle \langle \psi_1| + p_2 |\psi_2\rangle \langle \psi_2|$$

# Indistinguishability

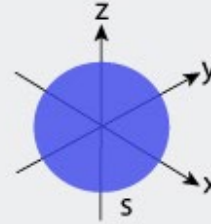


1 or 2?

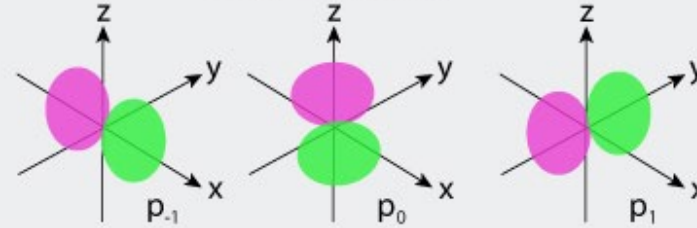
# Atomic Orbitals

Fermi  
exclusion  
principle

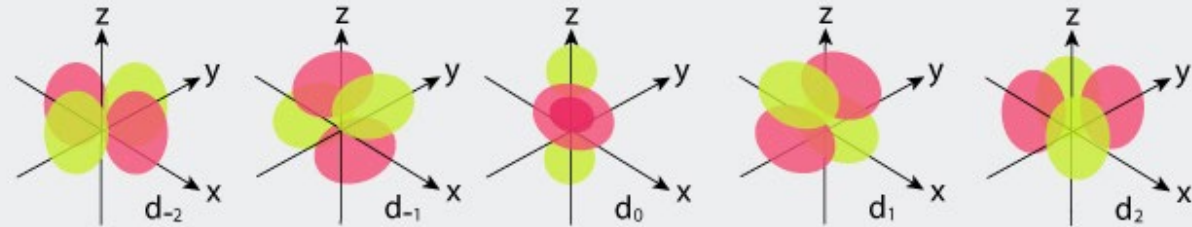
## 1. s-orbital



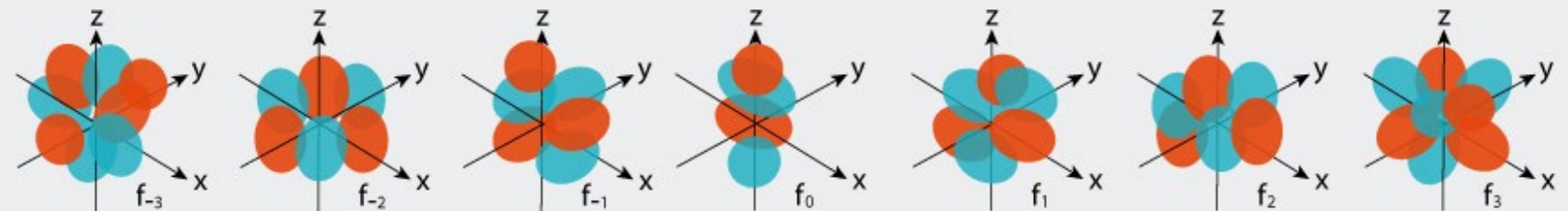
## 2. p-orbital



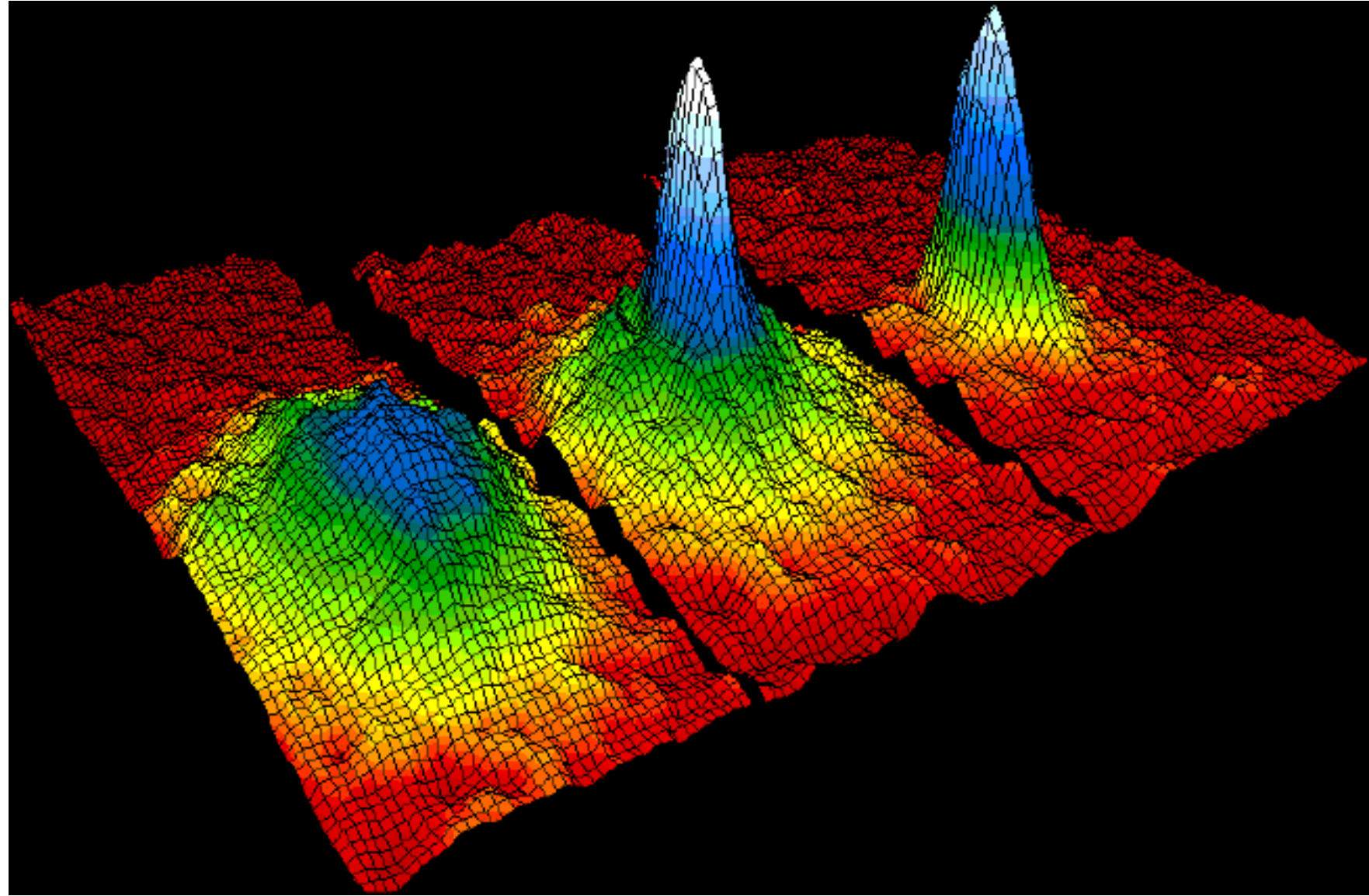
## 3. d-orbital



## 4. f-orbital

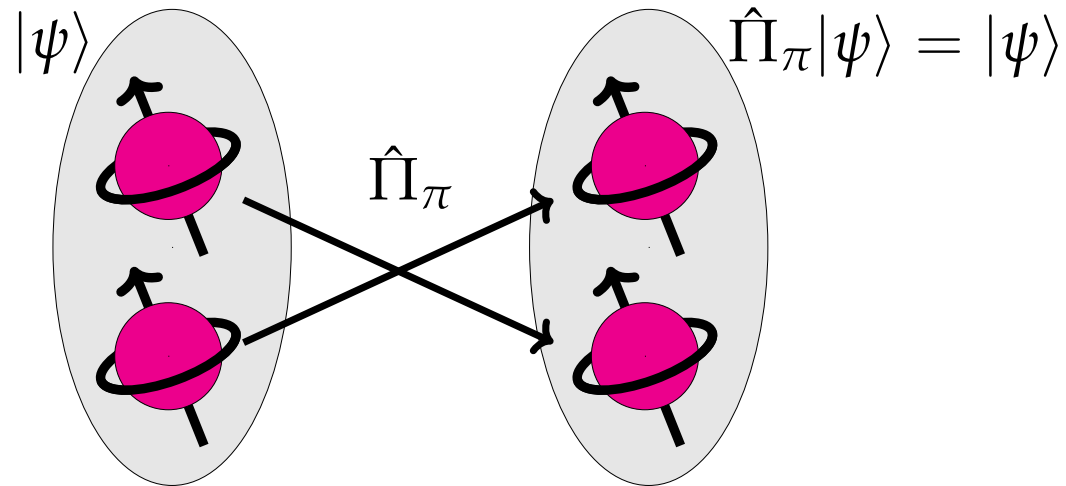


Bose-  
einstein  
condensate

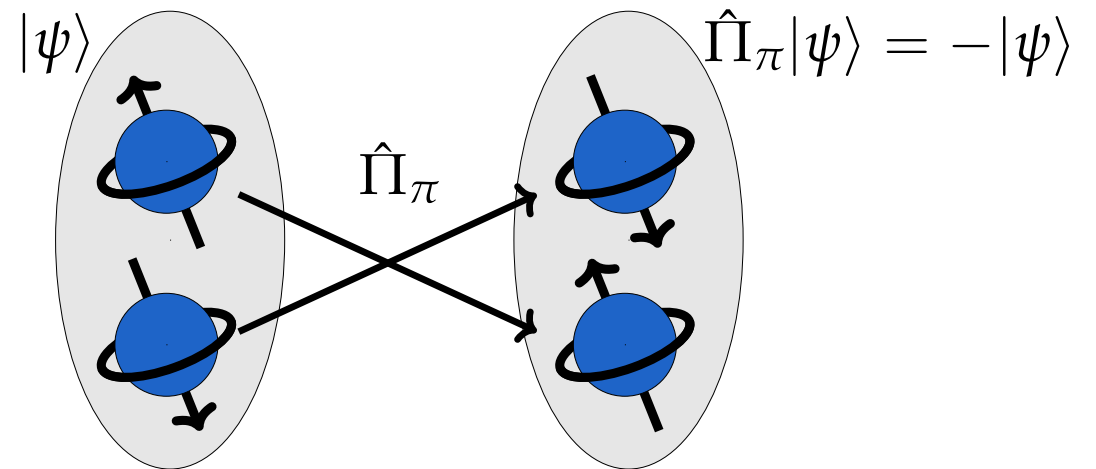


# Identical particles

**Bosons**



**Fermions**





Why do we have these two categories?

Curious  
guy



Symmetrisation postulate!

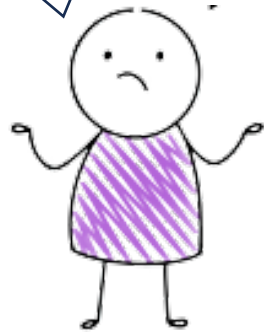
When a system is made up of several identical particles, only certain states in its state space can describe its physical states. Physical states are, depending on the nature of the identical particles, either symmetric or antisymmetric with respect to permutation of these particles. Those particles for which the physical states are symmetric are bosons and those for which they're antisymmetric, fermions

QM guy



Can we retrieve this from a symmetry argument?

Curious  
guy



Exchangeability?

Me



# Exchangeability on?

Classical framework

QM equivalent

Probability distribution

Density operator

Set of probability distributions

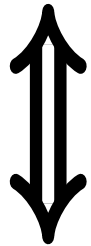
Set of density operators

Set of desirable gambles

Set of desirable measurements

# Set of density operators

Closed and convex



Up to boundary behaviour

# Set of desirable measurements

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<https://doi.org/10.1007/s10701-021-00499-w>



## The Weirdness Theorem and the Origin of Quantum Paradoxes

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### Abstract

We argue that there is a simple, unique, reason for all quantum paradoxes, and that such a reason is not uniquely related to quantum theory. It is rather a mathematical question that arises at the intersection of logic, probability, and computation. We give our ‘weirdness theorem’ that characterises the conditions under which the weirdness will show up. It shows that whenever logic has bounds due to the algorithmic nature of its tasks, then weirdness arises in the special form of negative probabilities or non-classical evaluation functionals. Weirdness is not logical inconsistency, however. It is only the expression of the clash between an unbounded and a bounded view of computation in logic. We discuss the implication of these results for quantum mechanics, arguing in particular that its interpretation should ultimately be computational rather than exclusively physical. We develop in addition a probabilistic theory in the real numbers that exhibits the phenomenon of entanglement, thus concretely showing that the latter is not specific to quantum mechanics.

# Set of desirable measurements $\mathcal{D}$

## Decision theoretic approach

## probabilities

### Decision-theoretic postulate 1

Let  $\mathcal{E}$  be the eigenspace corresponding to an eigenvalue  $\lambda$  of the measurement operator  $\hat{A} \in \mathcal{A}_h$ . Then  $u_{\hat{A}}(|a\rangle) = \lambda$  for all  $|a\rangle \in \mathcal{E}$  with  $\langle a|a\rangle = 1$ .

### Decision-theoretic postulate 2

Let  $\hat{A} \in \mathcal{A}_h^n$  be an operator and let  $\mathcal{B} = \{|a_1\rangle, |a_2\rangle, \dots, |a_n\rangle\}$  be a orthonormal basis of eigenkets of  $\hat{A}$ , where the eigenket  $|a_i\rangle$  corresponds to eigenvalue  $\lambda_i$  for all  $i \in \{1, 2, \dots, n\}$ . Let  $\pi : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$  be any permutation of the indices. Now let  $\hat{B} \in \mathcal{A}_h^n$  be the operator with the same eigenkets  $|a_1\rangle, |a_2\rangle, \dots, |a_n\rangle$ , but now corresponding to the respective eigenvalues  $\lambda_{\pi(1)}, \lambda_{\pi(2)}, \dots, \lambda_{\pi(n)}$ . Consider the normalised kets  $|\psi_{\hat{A}}\rangle, |\psi_{\hat{B}}\rangle \in \tilde{\mathcal{H}}$  described by

$$|\psi_{\hat{A}}\rangle = \sum_{i=1}^n \alpha_i |a_i\rangle \text{ and } |\psi_{\hat{B}}\rangle = \sum_{i=1}^n \alpha_{\pi(i)} |a_i\rangle.$$

Then

$$u_{\hat{A}}(|\psi_{\hat{A}}\rangle) = u_{\hat{B}}(|\psi_{\hat{B}}\rangle).$$

### Decision-theoretic postulate 3

Let  $\hat{A} \in \mathcal{A}_h^n$  be an operator with eigenvalue  $\lambda_i$  corresponding to eigenspace  $\mathcal{E}_i$  with  $i \in \{1, 2, \dots, p\}$ . Now take  $\hat{B} \in \mathcal{A}_h^{n+m}$  as the operator with eigenvalues  $\lambda_i$  corresponding to eigenspaces  $\mathcal{E}_i \otimes \mathcal{H}^m$  with  $i \in \{1, 2, \dots, p\}$ . For all  $|\psi\rangle \in \tilde{\mathcal{H}}^n$  and  $|\phi\rangle \in \tilde{\mathcal{H}}^m$ , we then have that  $u_{\hat{A}}(|\psi\rangle) = u_{\hat{B}}(|\psi\rangle \otimes |\phi\rangle)$ .

### Decision-theoretic postulate 4

If  $\hat{A}, \hat{B} \in \mathcal{A}_h^n$  are commuting operators, then  $u_{\hat{A}+\hat{B}}(|\psi\rangle) = u_{\hat{A}}(|\psi\rangle) + u_{\hat{B}}(|\psi\rangle)$  for all  $|\psi\rangle \in \tilde{\mathcal{H}}$ .

### Decision-theoretic postulate 5

The utility function is continuous: let  $\hat{A} \in \mathcal{A}_h^n$  be any operator and  $|\psi_1\rangle, |\psi_2\rangle, \dots \in \tilde{\mathcal{H}}$  be any sequence of kets, then

$$\lim_{i \rightarrow +\infty} |\psi_i\rangle = |\psi\rangle \implies \lim_{i \rightarrow +\infty} u_{\hat{A}}(|\psi_i\rangle) = u_{\hat{A}}(|\psi\rangle).$$

# Exchangeability through indifference

imate Reasoning

journal homepage: [www.elsevier.com/locate/ijar](http://www.elsevier.com/locate/ijar)



## Exchangeability and sets of desirable gambles

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### ABSTRACT

Sets of desirable gambles constitute a quite general type of uncertainty model with an interesting geometrical interpretation. We give a general discussion of such models and their rationality criteria. We study exchangeability assessments for them, and prove counterparts of de Finetti's Finite and Infinite Representation Theorems. We show that the finite representation in terms of count vectors has a very nice geometrical interpretation, and

Set of desirable measurements  $\mathcal{D}$  exchangeable if

$$\mathcal{D} + \mathcal{I} \subseteq \mathcal{D}$$

with  $\mathcal{I} = \{\hat{A} - \hat{\Pi}_\pi^\dagger \hat{A} \hat{\Pi}_\pi : \hat{A} \in \mathcal{H}, \pi \in \mathbb{P}\}$

up to boundary behaviour



Set of density operators

# Set of density operators

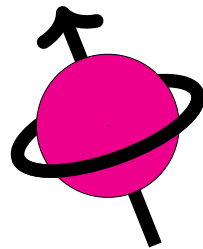
Permutation invariant

$$\hat{\rho} = \hat{\Pi}_\pi \hat{\rho} \hat{\Pi}_\pi^\dagger \text{ for all } \pi \in \mathbb{P}$$

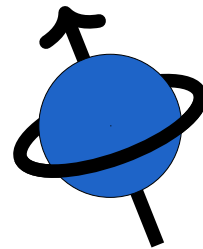


Probabilistic mixing of bosonic, fermionic and para-particle density operator

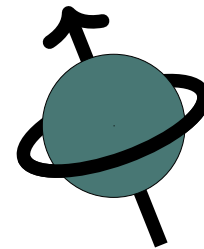
$$\hat{\rho} = p_s \hat{\rho}_s + p_a \hat{\rho}_a + p_o \hat{\rho}_o$$



or



or



possible



# Questions?

What are these para-particles?

Can we single out bosons and fermions?

Can we use count vectors?

Keano De Vos  
Gert de Cooman  
Jasper De Bock @UGent.be

## Indistinguishability through Exchangeability in Quantum Mechanics?

**State**  $|\psi\rangle$ : element of a finite dimensional Hilbert space  $\mathcal{H}$

**Measurement**  $\hat{A}$ : Hermitian operator on  $\mathcal{H}$

**Possible outcomes**: the eigenvalues  $\lambda$  of  $\hat{A}$

**Utility function**

$\hat{A} \rightarrow u_{\hat{A}}(|\psi\rangle) = \langle \psi | \hat{A} | \psi \rangle$

Possible motivations:  
1) Decision theoretic postulates  
2) Expected outcome of  $\hat{A}$  given Born's probabilistic rule

**Set  $\mathcal{D}$  of desirable measurements**

A measurement  $\hat{A}$  is desirable if you prefer the uncertain utility  $u_{\hat{A}}(|\Psi\rangle)$  to receiving nothing. This set is coherent if:

- D1.  $\hat{0} \notin \mathcal{D}$
- D2.  $\hat{A}_i \in \mathcal{D} \Rightarrow \hat{A}_i \in \mathcal{D}$  [strictness]
- D3.  $\hat{A}, \hat{B} \in \mathcal{D} \Rightarrow \hat{A} + \hat{B} \in \mathcal{D}$  [accepting sure gain]
- D4.  $\hat{A} \in \mathcal{D} \Rightarrow \lambda \hat{A} \in \mathcal{D}$  [positive scaling]

How can we model Your uncertainty about  $|\Psi\rangle$ ?

**Density operators**

- Density operator  $\rho := \sum_{i=1}^n p_i |\psi_i\rangle \langle \psi_i|$  with probability mass function  $p_1, \dots, p_n$  over possible states  $|\psi_1\rangle, \dots, |\psi_n\rangle$ .
- Expected outcome of  $\hat{A}$  is  $E_{\rho}(\hat{A}) = \text{Tr}(\rho \hat{A})$ .

Adding imprecision

**Credal set**

Set of density operators

$\mathcal{C}_{\mathcal{D}} = \{ \rho : (\forall \hat{A} \in \mathcal{D}) \text{Tr}(\rho \hat{A}) \geq 0 \}$

Up to boundary behaviour

### What changes when the particles in a system are indistinguishable?

**Standard approach: Symmetrisation postulate**

When a system is made up of several identical particles, only certain states in its state space can describe its physical states. Physical states are, depending on the nature of the identical particles, either symmetric or antisymmetric with respect to permutation of these particles. Those particles for which the physical states are symmetric are **bosons**, and those for which they are antisymmetric, **fermions**.

**Bosons (Symmetric particles)**

State  $|\psi\rangle \in \mathcal{H}_s; \hat{\Pi}_{\pi}|\psi\rangle = |\psi\rangle$  for all permutations  $\pi$ .

Bosonic density operator  $\hat{\rho}_s; \hat{\rho}_s = \hat{\Pi}_{\pi} \hat{\rho}_s = \hat{\rho}_s \hat{\Pi}_{\pi}^{\dagger}$  for all  $\pi$ .

**Fermions (Antisymmetric particles)**

State  $|\psi\rangle \in \mathcal{H}_a; \hat{\Pi}_{\pi}|\psi\rangle = \text{sgn}(\pi)|\psi\rangle$  for all permutations  $\pi$ .

Fermionic density operator  $\hat{\rho}_a; \hat{\rho}_a = \text{sgn}(\pi) \hat{\Pi}_{\pi} \hat{\rho}_a = \text{sgn}(\pi) \hat{\rho}_a \hat{\Pi}_{\pi}^{\dagger}$  for all  $\pi$ .

**Para-particles (HUUH?)**

These obey only the weaker permutation symmetry  $\hat{\rho}_p = \hat{\Pi}_{\pi} \hat{\rho}_p \hat{\Pi}_{\pi}^{\dagger}$ . As a collection of para-particles cannot be distinguished from standard particles, their existence has neither been confirmed nor excluded. In fact, some look to para-statistics in order to explain some properties of dark matter.

$\hat{\rho}_p = \frac{1}{3}(|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| + |\psi_3\rangle\langle\psi_3|)$

### Idea: indistinguishability through indifference

**Exchangeability**

Indistinguishable particles  $\Rightarrow$  order should not matter. Indifferent between receiving  $u_{\hat{A}}(|\Psi\rangle)$  and  $u_{\hat{A}}(\hat{\Pi}_{\pi}|\Psi\rangle)$ , or equivalently between executing  $\hat{A} - \hat{\Pi}_{\pi}^{\dagger} \hat{A} \hat{\Pi}_{\pi}$  and  $\hat{0}$ .

$\mathcal{S} = \{ \hat{A} - \hat{\Pi}_{\pi}^{\dagger} \hat{A} \hat{\Pi}_{\pi}; \hat{A} \in \mathcal{H}, \pi \in \mathbb{P} \}$

**Exchangeable set of density operators**

Every density operator  $\rho$  in the credal set  $\mathcal{C}_{\mathcal{S}}$  is permutation invariant:  $\rho = \hat{\Pi}_{\pi} \rho \hat{\Pi}_{\pi}^{\dagger}$  for all  $\pi \in \mathbb{P}$ . Equivalently, these density operators can be written as the probabilistic mixing of a bosonic, a fermionic and a para-particle density operator.

$\hat{\rho} = p_s \hat{\rho}_s + p_f \hat{\rho}_f + p_p \hat{\rho}_p$

still possible

**Strong +-symmetry**

How can we single out the assessment that the particles are symmetric,  $\ast = s$ , or antisymmetric,  $\ast = a$ ? The symmetry operator needs to break the quadratic symmetry:

$S_{\ast}^{\pm}(\hat{A}) := \frac{\text{sgn}(\pi)}{2} (\hat{\Pi}_{\pi}^{\dagger} \hat{A} + \hat{A} \hat{\Pi}_{\pi})$ , for  $\pi \in \mathbb{P}$ .

Indifferent between  $\hat{A} - S_{\ast}^{\pm}(\hat{A})$  and  $\hat{0}$ .

$\mathcal{S}^{\ast} := \{ \hat{A} - S_{\ast}^{\pm}(\hat{A}); \hat{A} \in \mathcal{H}, \pi \in \mathbb{P} \}$

**Strong +-symmetric set of density operators**

Every density operator  $\rho$  in the credal set  $\mathcal{C}_{\mathcal{S}^{\ast}}$  satisfies  $\rho = \text{sgn}(\pi) \hat{\Pi}_{\pi} \rho = \text{sgn}(\pi) \hat{\rho} \hat{\Pi}_{\pi}^{\dagger}$ .

$\ast = s$  or  $\ast = a$

**Second quantisation**

There is a lot of redundancy in the modelling of fermions and bosons. In quantum mechanics, this is solved by considering second quantisation. This approach considers occupation numbers and is completely similar to the count vector approach in imprecise probabilities. In fact, we proved that we can represent strong +-symmetric sets of desirable measurements on the whole  $\mathcal{X}$ , by smaller dimensional sets of desirable measurements on the subspace of count vectors.

¿¿¿Flip quiz???

What kind of particle is not modelled through strong +-symmetry?