A Pointfree Approach to Measurability and Statistical Models

Serafina Lapenta
joint work with A. Di Nola and G. Lenzi

University of Salerno

ISIPTA 2023

The problem

The problem

Rota's Fubini lectures

In his 1998 Fubini Lectures, Rota discusses twelve problems in probability that no one likes to bring up.

G–C. Rota, Twelve problems in probability no one likes to bring up, in: The Fubini Lectures, Torino, 3-5 June, 1998, in: H. Crapo, D. Senato (Eds.), Algebraic Combinatorics and Computer Science, a Tribute to Gian-Carlo Rota, Springer, Milan, 2001, pp.57–93

The problem

Rota's Fubini lectures

In his 1998 Fubini Lectures, Rota discusses twelve problems in probability that no one likes to bring up.

G–C. Rota, Twelve problems in probability no one likes to bring up, in: The Fubini Lectures, Torino, 3-5 June, 1998, in: H. Crapo, D. Senato (Eds.), Algebraic Combinatorics and Computer Science, a Tribute to Gian-Carlo Rota, Springer, Milan, 2001, pp.57–93

The first problem

I will lay my cards on the table: a revision of the notion of a sample space is my ultimate concern.

[...]

The problem

[...]

Pointless probability deals with an abstract Boolean σ-algebra Π and with a real-valued function defined on Π which imitates the definition of probability. The problem is to define an algebra of random variables. The setup is not as artificial as it may appear. Among probabilists, mention of sample points in an argument has always been bad form. A fully probabilistic argument must be pointless.

The problem

[...]

Pointless probability deals with an abstract Boolean σ-algebra Π and with a real-valued function defined on Π which imitates the definition of probability. The problem is to define an algebra of random variables. The setup is not as artificial as it may appear. Among probabilists, mention of sample points in an argument has always been bad form. A fully probabilistic argument must be pointless.

Nelson's point of view

Addition and multiplication are the bare minimum needed for any algebraic approach to probability theory!

E. Nelson, Radically elementary probability theory, Ann. Math. Stud. 117 (1987).

An Answer using Boolean algebras

By Stone duality every boolean algebra A is an algebra of continuous $\{0,1\}$-valued random variables, or yes-no events, defined over the maximal spectral space $\mathrm{Max}\,(A)$.

An Answer using Boolean algebras

By Stone duality every boolean algebra A is an algebra of continuous $\{0, 1\}$-valued random variables, or yes-no events, defined over the maximal spectral space $\mathsf{Max}(A)$.

Using (AF) C^*-algebras, Boolean algebras have a hidden ring-theoretic structure, as advocated by E. Nelson.

An Answer using Boolean algebras

By Stone duality every boolean algebra A is an algebra of continuous $\{0, 1\}$-valued random variables, or yes-no events, defined over the maximal spectral space $\mathrm{Max}\,(A)$.

Using (AF) C^*-algebras, Boolean algebras have a hidden ring-theoretic structure, as advocated by E. Nelson.

Furthermore, via de Finetti's Dutch book theorem, a (logical-)algebraic definition of probability can be given on Boolean algebras.

An Answer using Boolean algebras

By Stone duality every boolean algebra A is an algebra of continuous $\{0, 1\}$-valued random variables, or yes-no events, defined over the maximal spectral space $\mathsf{Max}\,(A)$.

Using (AF) C^*-algebras, Boolean algebras have a hidden ring-theoretic structure, as advocated by E. Nelson.

Furthermore, via de Finetti's Dutch book theorem, a (logical-)algebraic definition of probability can be given on Boolean algebras.

The limitation

The measures that can be considered when dealing with Boolean algebras are only defined on the Borel sets of the totally disconnected compact Hausdorff space $\mathsf{Max}\,(A)$.

This excludes too many measure spaces that are of interest for probabilists!

An Answer for continuous random variables - D. Mundici

An Answer for continuous random variables - D. Mundici

MV-algebras are a generalization of Boolean algebras that were defined as the algebraic semantics for the infinite valued Łukasiewicz logic.

An Answer for continuous random variables - D. Mundici

MV-algebras are a generalization of Boolean algebras that were defined as the algebraic semantics for the infinite valued Łukasiewicz logic.

Once again, using (AF) C^*-algebras, Boolean algebras have a hidden ring-theoretic structure, as advocated by E. Nelson.

An Answer for continuous random variables - D. Mundici

MV-algebras are a generalization of Boolean algebras that were defined as the algebraic semantics for the infinite valued Łukasiewicz logic.

Once again, using (AF) C^*-algebras, Boolean algebras have a hidden ring-theoretic structure, as advocated by E. Nelson.

Furthermore, D. Mundici showed in a series of papers how to define a notion of algebraic probability measure on MV-algebras and how to extend de Finetti's Dutch book theorem.

An Answer for continuous random variables - D. Mundici

MV-algebras are a generalization of Boolean algebras that were defined as the algebraic semantics for the infinite valued Łukasiewicz logic.

Once again, using (AF) C^*-algebras, Boolean algebras have a hidden ring-theoretic structure, as advocated by E. Nelson.

Furthermore, D. Mundici showed in a series of papers how to define a notion of algebraic probability measure on MV-algebras and how to extend de Finetti's Dutch book theorem.

Indeed the closed compact set $\mathcal{S}(A) \subseteq [0,1]^A$ of finitely additive probability measures(=states) on A coincides with the set of $[0,1]$-valued functions on A whose finite restrictions are coherent à la de Finetti's.

An Answer for continuous random variables - D. Mundici

Theorem (Mundici, 2021)

For every Kolmogorov probability space $(\Omega, \mathcal{BO}(\Omega), P)$*, with* $\mathcal{BO}(\Omega)$ *the* σ*-algebra of Borel sets of a compact Hausdorff space* Ω*, and* P *a regular probability measure on* $\mathcal{BO}(\Omega)$*, there is an MV-algebra* A *and a state* $s \in \mathcal{S}(A)$ *such that*

$$(\Omega, \mathcal{BO}(\Omega), P) \simeq (\mathrm{Max}\,(A)\,, \mathcal{BO}(\mathrm{Max}\,(A)), s).$$

D. Mundici, Rota's Fubini lectures: The first problem, Advances in Applied Mathematics, 125: 102153, 2021.

How to go beyond continuity to measurable functions

How to go beyond continuity to measurable functions

MV-algebras closed under scalar operation and countable suprema, endowed with σ-homomorphisms of Riesz MV-algebras.

How to go beyond continuity to measurable functions

MV-algebras closed under scalar operation and countable suprema, endowed with σ-homomorphisms of Riesz MV-algebras.

This are an infinitary variety called $\mathbf{RMV}_\sigma$ and we can think of them as algebras of continuous functions, closed under countable suprema.

How to go beyond continuity to measurable functions

MV-algebras closed under scalar operation and countable suprema, endowed with σ-homomorphisms of Riesz MV-algebras.

This are an infinitary variety called RMV_σ and we can think of them as algebras of continuous functions, closed under countable suprema. Indeed,

$$A \in \mathsf{RMV}_\sigma \quad \Leftrightarrow \quad A = C(X),$$

where X is a basically disconnected compact Hausdorff space and functions in $C(X)$ are continuous are $[0,1]$-valued.

How to go beyond continuity to measurable functions

MV-algebras closed under scalar operation and countable suprema, endowed with σ-homomorphisms of Riesz MV-algebras.

This are an infinitary variety called RMV_σ and we can think of them as algebras of continuous functions, closed under countable suprema. Indeed,

$$A \in \mathsf{RMV}_\sigma \quad \Leftrightarrow \quad A = C(X),$$

where X is a basically disconnected compact Hausdorff space and functions in $C(X)$ are continuous are $[0,1]$-valued.

Free algebras in RMV_σ have been characterized as algebras of measurable functions. If $\kappa \leq \omega$, then the free κ-generated algebra in RMV_σ is the algebra of all Borel measurable functions $\mathsf{Borel}([0,1]^\kappa)$.

A privileged subclass

An algebra $A \in \mathsf{RMV}_\sigma$ is σ-semisimple if, and only if, $A \in ISP([0,1])$.

A privileged subclass

An algebra $A \in \mathsf{RMV}_\sigma$ is σ-semisimple if, and only if, $A \in ISP([0,1])$.

If $\mathcal{B}$ is any σ-algebra on any set B,

$$\mathsf{Meas}(B,[0,1]) = \{f\colon B \to [0,1] \mid f \text{ is a measurable function}\},$$

where we take the unit interval endowed with the σ-algebra of its Borel subsets.

A privileged subclass

An algebra $A \in \mathrm{RMV}_\sigma$ is σ-semisimple if, and only if, $A \in ISP([0,1])$.

If $\mathcal{B}$ is any σ-algebra on any set B,

$$\mathrm{Meas}(B,[0,1]) = \{f\colon B \to [0,1] \mid f \text{ is a measurable function}\},$$

where we take the unit interval endowed with the σ-algebra of its Borel subsets.

Theorem (Di Nola, L., Lenzi)

A σ-complete Riesz MV-algebra A is isomorphic to a Riesz MV-algebra of the form $\mathrm{Meas}(B,[0,1])$ if and only if A is σ-semisimple.

Why this setting?

Classical probability	Non-classical probability
probability measure	state
σ-algebra of events	
event E (or χ_E)	
random variable	

Why this setting?

Classical probability	Non-classical probability
probability measure	state
σ-algebra of events	algebra in RMV_σ
event E (or χ_E)	
random variable	

Why this setting?

Classical probability	Non-classical probability
probability measure	state
σ-algebra of events	algebra in RMV_σ
event E (or χ_E)	function $f \in \mathsf{Borel}([0,1]^\kappa)$
random variable	

Why this setting?

Classical probability	Non-classical probability
probability measure	state
σ-algebra of events	algebra in RMV_σ
event E (or χ_E)	function $f \in \mathrm{Borel}([0,1]^\kappa)$
random variable	?

Why this setting?

Classical probability	Non-classical probability
probability measure	state
σ-algebra of events	algebra in RMV_σ
event E (or χ_E)	function $f \in \mathsf{Borel}([0,1]^\kappa)$
random variable	?

Classical vs Generalized Random Variables

We argue that in this algebraic framework we can at the same time answer to Rota's problem and generalize random variables.

Di Nola A., Dvurečenskij A., Lapenta S., An approach to stochastic processes via non-classical logic, Annals of Pure and Applied Logic 172(9) 2021.

On Rota's problem

Mundici proved that moving from Boolean Algebras to (certain) MV-algebras, allows to describe algebraically all continuous random variables on compact Hausdorff spaces.

On Rota's problem

Mundici proved that moving from Boolean Algebras to (certain) MV-algebras, allows to describe algebraically all continuous random variables on compact Hausdorff spaces.

Moving from (certain) MV-algebras to σ-continuous Riesz MV-algebras it seems, at first, that one is restricting again to certain compact Hausdorff space.

On Rota's problem

Mundici proved that moving from Boolean Algebras to (certain) MV-algebras, allows to describe algebraically all continuous random variables on compact Hausdorff spaces.

Moving from (certain) MV-algebras to σ-continuous Riesz MV-algebras it seems, at first, that one is restricting again to certain compact Hausdorff space.

But, restricting to σ-semisimple algebras we are able to describe all measurable functions $f\colon (B,\mathcal{B}) \to ([0,1],\mathcal{BO}([0,1])$.

On Rota's problem

Mundici proved that moving from Boolean Algebras to (certain) MV-algebras, allows to describe algebraically all continuous random variables on compact Hausdorff spaces.

Moving from (certain) MV-algebras to σ-continuous Riesz MV-algebras it seems, at first, that one is restricting again to certain compact Hausdorff space.

But, restricting to σ-semisimple algebras we are able to describe all measurable functions $f\colon (B, \mathcal{B}) \to ([0,1], \mathcal{BO}([0,1]))$.

In the paper we also show how we take this answer to Rota's problem a step further, and translate all results in the language of frames and category theory.

On Rota's problem

Mundici proved that moving from Boolean Algebras to (certain) MV-algebras, allows to describe algebraically all continuous random variables on compact Hausdorff spaces.

Moving from (certain) MV-algebras to σ-continuous Riesz MV-algebras it seems, at first, that one is restricting again to certain compact Hausdorff space.

But, restricting to σ-semisimple algebras we are able to describe all measurable functions $f\colon (B, \mathcal{B}) \to ([0,1], \mathcal{BO}([0,1]))$.

In the paper we also show how we take this answer to Rota's problem a step further, and translate all results in the language of frames and category theory.

We also tackle the notion of logico-algebraic statistical model in this new framework.

Thank you!