Neighbourhood models induced by the Euclidean distance and the Kullback-Leibler divergence

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Asturias: Natural Paradise

Asturias
paraiso natural

IP group in Oviedo

IP researchers

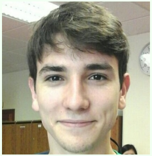

E.Miranda


I.Montes


J.J.Salamanca


R.Pérez

Young people


J.Álvarez


D.Nieto


M.Porrón


A.Presa

Close reseracher(s)


A.Bouchet

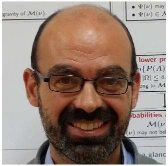

S.Díaz


I.Mariñas


S.Montes

Distortion models

S.Destercke



E.Miranda

P.R.Alonso

- Unifying neighbourhood and distortion models: Part I. Montes, Miranda, Destercke. IJGS 2020.
- Unifying neighbourhood and distortion models: Part II. Montes, Miranda, Destercke. IJGS 2020.
- Processing distortion models: a comparative study. Destercke, Montes, Miranda. IJAR 2022.
- Distortion models for estimating human error probabilities. Alonso, Montes, Miranda. SS 2023.

Distortion models

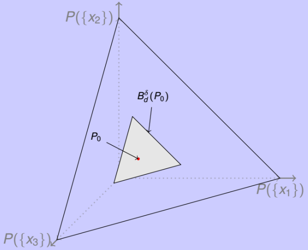
Ingredients

P_0: probability measure

δ: distortion parameter

d: distorting function

Example



The model

$$B_d^\delta(P_0) = \{P \mid d(P, P_0) \leq \delta\}$$

$$\underline{P}_d(f) = \inf\{P(f) \mid P \in B_d^\delta(P_0)\}$$

$$\mathcal{M}(\underline{P}_d) = \{P \mid P(f) \geq \underline{P}_d(f)\ \forall f\}$$

d continuous and convex

Particular models

Pari Mutuel Model

$$\underline{P}_{PMM}(A) = \max\{0, (1+\delta)P_0(A) - \delta\}$$

Pelessoni et al., 2010
Montes et al., 2019
Walley, 1991

Linear vacuous model

$$\underline{P}_{LV}(A) = (1-\delta)P_0(A) \qquad A \neq \mathcal{X}$$

Huber, 1981
Walley, 1991

Constant odds ratio

$$\underline{P}_{COR}(A) = \frac{(1-\delta)P_0(A)}{1-\delta P_0(A)}$$

Benavoli and Zaffalon, 2013
Walley, 1991

Total Variation model

$$\underline{P}_{TV}(A) = \max\{0, P_0(A) - \delta\} \qquad A \neq \mathcal{X}$$

Montes et al., 2020
Herron et al., 1997

Vertical barrier models

$$\underline{P}_{VB}(A) = \max\{bP_0(A) + a, 0\} \qquad a \leq 0, b > 0, \; a + b \in (0,1)$$

Pelessoni et al., 2021
Corsato et al., 2019

Increasing transformation

$$\underline{P}(A) = g(P_0(A))$$

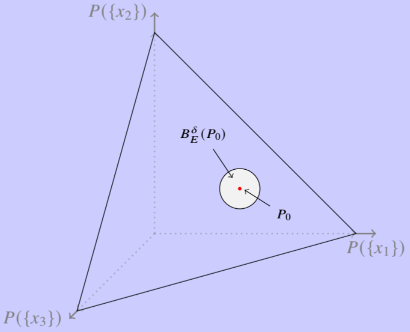
$g : [0,1] \to [0,1]$ increasing
$g(0) = 0, g(1) = 1$

Bronevich, 2005

Aim of the paper

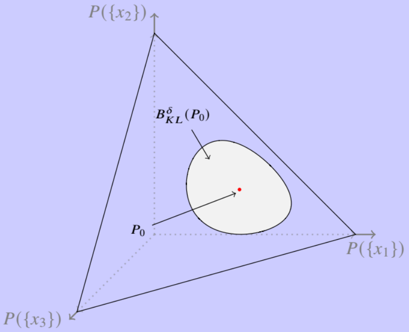
Euclidean model

$$d_E(P,Q) = \sqrt{\sum_{i=1}^{n}(p_i - q_i)^2}$$



Kullback-Leibler model

$$D_{KL}(P,Q) = \sum_{i=1}^{n} p_i \log\left(\frac{p_i}{q_i}\right)$$



Euclidean model

Expression for $\underline{P}_E$:

$$\underline{P}_E(f) = \textbf{??}$$

$$\underline{P}_E(A) = \textbf{??}$$

Extreme points:

$$\text{ext}\big(B_E^\delta(P_0)\big) = \textbf{??}$$

Properties:

2-monotonicity **??**

Complete-monotonicity **??**

Conditioning **??**

KL-model

Expression for $\underline{P}_{KL}$:

$$\underline{P}_{KL}(f) = \textbf{??}$$

$$\underline{P}_{KL}(A) = \textbf{??}$$

Extreme points:

$$\text{ext}\big(B^{\delta}_{KL}(P_0)\big) = \textbf{??}$$

Properties:

2-monotonicity **??**

Complete-monotonicity **??**

Conditioning **??**

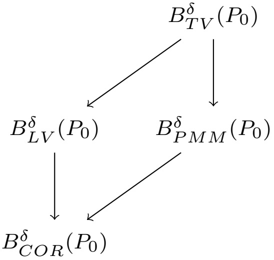
Comparison: properties

	2-monot.	∞-monot.	Prob.interval	Conditioning
Euclidean-model...				
...on gambles	??	??	??	??
...on events	??	??	??	??
KL-model...				
...on gambles	??	??	??	??
...on events	??	??	??	??
PMM	✓	✗	✓	✓
LV	✓	✓	✓	✓
COR...				
...on gambles	✗	✗	✗	✓
...on events	✓	✓	✗	✓
TV	✓	✗	✓	✓
Vertical Barrier models	✓	✗	✗	✓

Comparison: amount of imprecision

$\boxed{B^{\delta}_{KL}(P_0)}$ **??**

$\boxed{B^{\delta}_{E}(P_0)}$ **??**



Neighbourhood models induced by the Euclidean distance and the Kullback-Leibler divergence

Universidad de Oviedo

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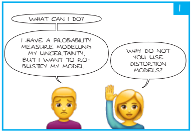
ISIPTA 2021



DISTORTION MODELS

1. INGREDIENTS

P_0: probability measure

δ: distortion parameter

d: distorting function

Assumptions:

$\mathcal{X}=\{x_1,\ldots,x_n\}$

$P_0(A)>0\ \forall A\neq\emptyset$

δ *"small enough"*

2. THE MODEL

$$B_d^\delta(P_0)=\{P \mid d(P,P_0)\leq\delta\}$$

$\downarrow$

$$\underline{P}_d(f)=\inf\{P(f)\mid P\in B_d^\delta(P_0)\}$$

$\downarrow$

$$\mathcal{M}(\underline{P}_d)=\{P\mid P(f)\geq\underline{P}_d(f)\ \forall f\}$$

d continuous and convex

3. PARTICULAR CASES

3.1 Known IP-models

$\underline{P}_{PMM}(A)=\max\{0,(1+\delta)P_0(A)-\delta\}$

$\underline{P}_{LV}(A)=(1-\delta)P_0(A)$ $\quad A\subset\mathcal{X}$

$\underline{P}_{COR}(A)=\frac{(1-\delta)P_0(A)}{1-\delta P_0(A)}$

3.2 Distance-based models

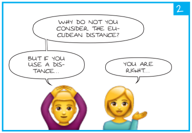
$\underline{P}_{TV}(A)=\max\{0,P_0(A)-\delta\}$ $\quad A\subset\mathcal{X}$

3.3 Vertical barrier models

$\underline{P}_{VB}(A)=\max\{bP_0(A)+a,0\}$ $\quad a\leq 0,\ b>0,\ a+b\in[0,1]$

3.4 Increasing transformations

$\underline{P}(A)=g(P_0(A))$ $\quad g:[0,1]\to[0,1]$ increasing, $g(0)=0,\ g(1)=1$



EUCLIDEAN MODEL

1. Euclidean distance

$$d_E(P,P_0)=\sqrt{\sum_{i=1}^n(p_i-p_i^0)^2}$$

d_E continuous and convex

2. Lower prevision

$$f=\sum_{i=1}^n a_iI_{x_i}\quad \overline{f}=\frac{1}{n}\sum_{i=1}^n a_i,\quad S_f^2=\frac{1}{n}\sum_{i=1}^n(a_i-\overline{f})^2$$

$$\underline{P}_E(f)=P_0(f)-\delta\sqrt{n}S_f$$

3. Lower probability

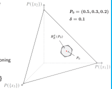
$$\underline{P}_E(A)=P_0(A)-\delta\sqrt{\frac{|A|(n-|A|)}{n}}\quad A\subset\mathcal{X}$$

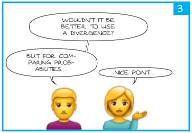
4. Properties

1) $\underline{P}_E$ is not 2-monotone…
2) …but it is 2-monotone on events
3) the model is not preserved under conditioning
4) extreme points:

$$\text{ext}(B_E^\delta(P_0))=\{P\mid d_E(P,P_0)=\delta\}$$

5. Example





KULLBACK-LEIBLER MODEL

1. KL-divergence

$$D_{KL}(P,P_0)=\sum_{i=1}^n p_i\log\left(\frac{p_i}{p_i^0}\right)$$

D_{KL} continuous and convex

2. Lower prevision

$$f=\sum_{i=1}^n a_iI_{x_i}$$

$$\underline{P}_{KL}(f)=\min_{\alpha_1,\ldots,\alpha_n}\sum_{i=1}^n a_i\alpha_ip_i^0$$

subject to

$$\sum_{i=1}^n\alpha_ip_i^0=1,\quad\sum_{i=1}^n\alpha_ip_i^0\log(\alpha_i)=\delta$$

3. Lower probability

$$\underline{P}_{KL}(A)=\alpha P_0(A)$$

where $\alpha\in[0,1]$ is the unique solution of

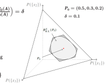
$$\alpha P_0(A)\log(\alpha)+(1-\alpha P_0(A))\log\left(\frac{1-\alpha P_0(A)}{1-P_0(A)}\right)=\delta$$

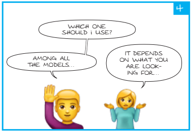
4. Properties

1) $\underline{P}_{KL}$ is not 2-monotone…
2) …but it is 2-monotone on events
3) the model is not preserved under conditioning
4) extreme points:

$$\text{ext}(B_{KL}^\delta(P_0))=\{P\mid D_{KL}(P,P_0)=\delta\}$$

5. Example



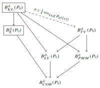


COMPARISON

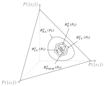
1. Properties

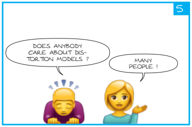
	2-monot.	∞-monot.	Prob.interval	Conditioning
Euclidean-model…				
…on gambles	✗	✗	✗	✗
…on events	✓	✗	✗	✗
KL-model…				
…on gambles	✗	✗	✗	✗
…on events	✓	✗	✗	✗
PMM	✓	✗	✓	✓
LV	✓	✓	✓	✓
COR…				
…on gambles	✗	✗	✗	✓
…on events	✓	✓	✗	✓
TV	✓	✗	✓	✓
Vertical Barrier models	✓	✗	✗	✓

2. Amount of imprecision



3. Example





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