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On Distinct Belief Functions in the D-S Theory

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Introduction

Dempster's combination rule is designed to combine distinct (independent) belief functions

- What constitutes distinct belief functions?
- What are distinct belief functions in directed graphical models
- What are distinct belief functions in undirected graphical models

1 Dempster's Definition of Distinct BFs

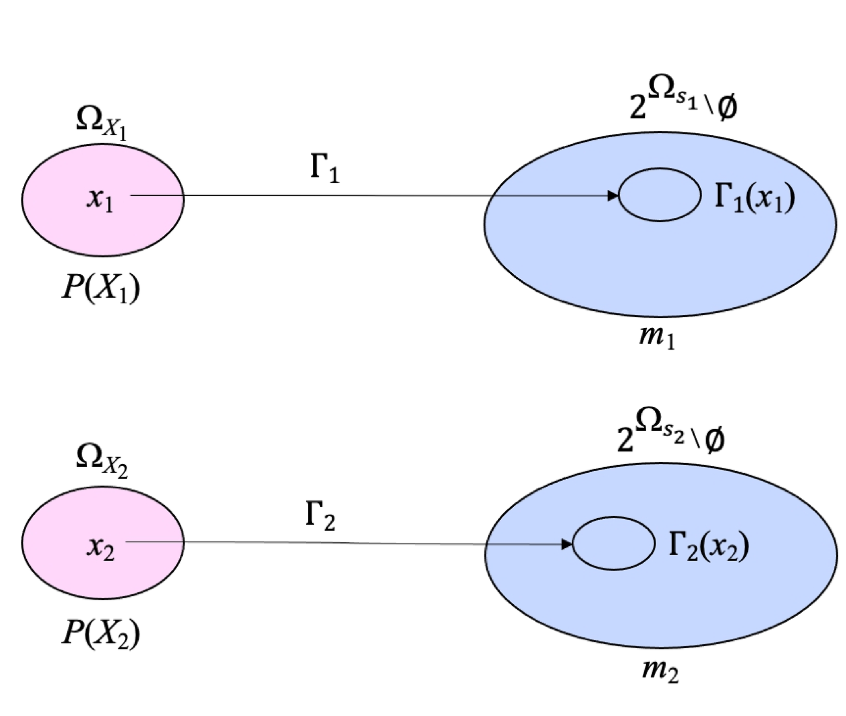

Figure 1: Dempster's multivalued mapping semantics

Definition 1 (Distinct belief functions (Dempster 1967))**.** *Consider the situation in Fig. 1. m_1 and m_2 are distinct iff X_1 and X_2 are independent.*

The essence of Dempster's definition is two BPAs are distinct iff there is no double-counting of non-idempotent knowledge.

2 Conditional Independence in the DS Theory

Definition 2 (Conditional independence (Shenoy 1994))**.** *Suppose $\mathcal{V}$ denotes the set of all variables, and suppose r, s, and t are disjoint subsets of $\mathcal{V}$. Suppose m is a joint BPA for $\mathcal{V}$. We say r and s are conditionally independent given t with respect to BPA m, written as $r \perp\!\!\!\perp_m s \,|\, t$, if and only if $m^{\downarrow r \cup s \cup t} = m_{r \cup t} \oplus m_{s \cup t}$, where $m_{r \cup t}$ is a BPA for $r \cup t$, $m_{s \cup t}$ is a BPA for $s \cup t$, and $m_{r \cup t}$ and $m_{s \cup t}$ are distinct.*

This definition generalizes the CI relation in probability theory (Dawid 1979). The definition of CI in Def. 2 satisfies the graphoid properties of probabilistic conditional independence (Pearl and Paz 1987).

3 Non-informative BFs

Definition 3 (Non-informative BFs)**.** *Suppose m_1 and m_2 are BPAs for s_1 and s_2, resp. We say m_1 and m_2 are mutually non-informative if $m_1^{\downarrow s_1 \cap s_2}$ and $m_2^{\downarrow s_1 \cap s_2}$ are vacuous BPAs for $s_1 \cap s_2$. If $\mathcal{M} = \{m_1, \ldots m_n\}$ is a set of BPAs, we say $\mathcal{M}$ is non-informative if every pair of BPAs in $\mathcal{M}$ is mutually non-informative.*

4 Directed Graphical Models in the DS Theory

For any node $X \in \mathcal{V}$, let $Pa_{G_d}(X)$ denote $\{Y \in \mathcal{V} : (Y, X) \in \mathcal{E}\}$, the parents of X in $G_d = (\mathcal{V}, \mathcal{E})$.

Definition 4 (BF directed graphical model)**.** *Suppose we have a directed acyclic graph $G_d = (\mathcal{V}, \mathcal{E})$ with n nodes in $\mathcal{V}$. A belief-function directed graphical model ($BFDGM$) is a pair $(G_d, \{m_1, \ldots, m_n\})$ such that BPA m_i associated with node X_i is a conditional BPA for X_i given $Pa_{G_d}(X_i)$, for $i = 1, \ldots, n$. A fundamental assumption of a BFDGM is that $m_1, \ldots, m_n$ are all distinct, and the joint BPA m for $\mathcal{V}$ associated with the model is given by*

$$m = \bigoplus_{i=1}^{n} m_i. \tag{1}$$

Some comments about Def. 4:

1. The assumption in Def. 4 that all conditionals are distinct allows the combination in Eq. (1).
2. Given m, the joint BPA for $\mathcal{V}$ as defined in Eq. (1), it follows from Def. 2 that the following CI relations hold. Suppose $(X_1, \ldots, X_n)$ is a topological sequence associated with BFDGM $(G_d, \{m_1, \ldots, m_n\})$. Then for each X_i, $i = 2, \ldots, n$, $X_i \perp\!\!\!\perp_m (\{X_1, \ldots X_{i-1}\} \setminus Pa_{G_d}(X_i)) \mid Pa_{G_d}(X_i)$.
3. $m_1, \ldots m_n$ are distinct iff each m_i is a conditional for X_i given $Pa_{G_u}(X_i)$ for $i = 1, \ldots, n$, and the CI assumptions of the model are all valid.

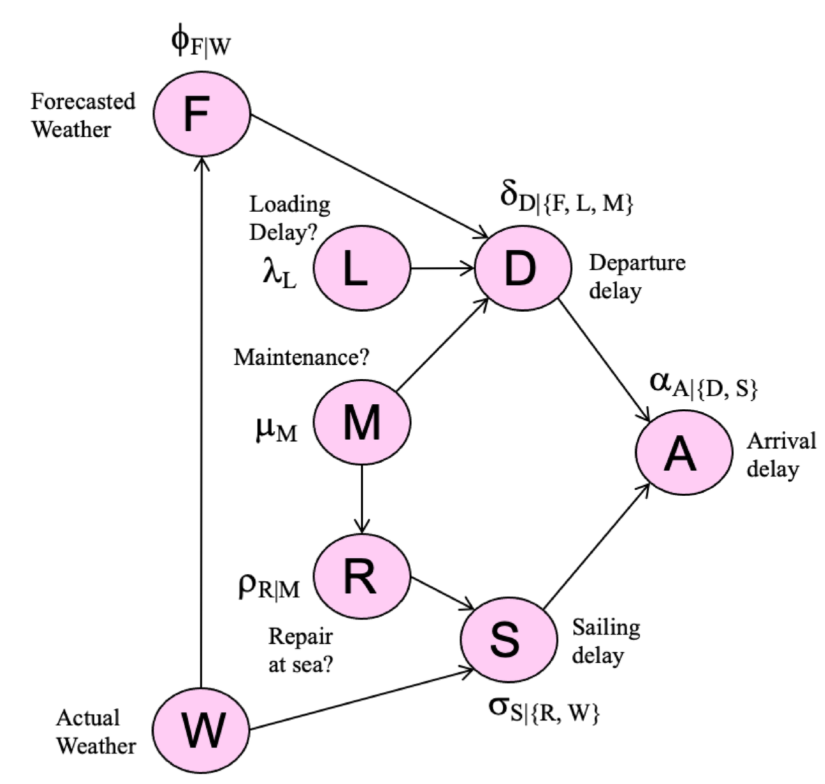

Figure 2: Captain's problem

5 Undirected Graphical Models in the DS Theory

Given a variable $X \in \mathcal{V}$, the Markov blanket of X, denoted by $MB_{G_u}(X)$, is $\{Y \in \mathcal{V} : \{X, Y\} \in E\}$. A *clique* in $G_u = (\mathcal{V}, \mathcal{E})$ is a maximal complete subset of $\mathcal{V}$.

Definition 5 (BF undirected graphical model)**.** *A belief-function undirected graphical model (BFUGM) is $\big(G_u = (\mathcal{V}, \mathcal{E}), \{m_1, \ldots, m_k\}\big)$, where G_u is an undirected graph with cliques $r_1, \ldots, r_k$, and for each $i = 1, \ldots, k$, m_i is a BPA for r_i. A fundamental assumption of a BFUGM is that the BPAs are all distinct. Thus, a belief-function undirected graphical model corresponds to the joint BPA m for $\mathcal{V}$ defined as follows:*

$$m = \bigoplus_{i=1}^{k} m_i, \tag{2}$$

assuming that m as defined in Eq. (2) is a well-defined BPA, i.e., the normalization constant K in Dempster's combination is non-zero.

Some comments about Def. 5.

1. The assumption in Def. 5 that the BPAs are all distinct allows the combination in Eq. (2).
2. Given m, the joint BPA for $\mathcal{V}$, it follows from Def. 5 that the following CI relations hold. For each $X \in \mathcal{V}$, $X \perp\!\!\!\perp_m (\mathcal{V} \setminus (MB_{G_u}(X) \cup \{X\}) \mid MB_{G_u}(X)$.
3. $m_1, \ldots, m_k$ are distinct if $\{m_1, \ldots, m_k\}$ is a non-informative set of BPAs, and the CI assumptions of the model are all valid.

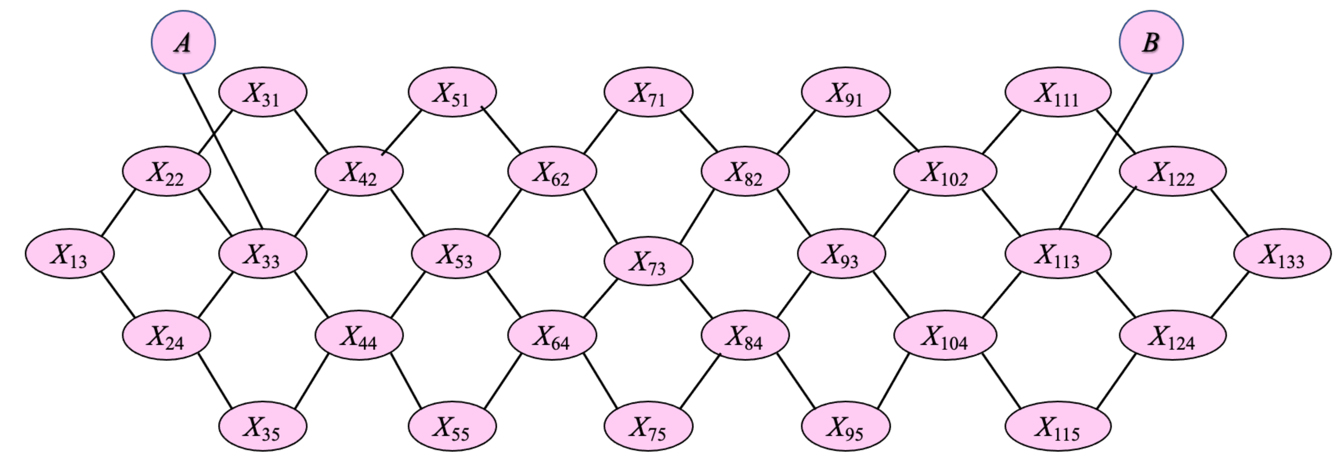

Figure 3: Communication network problem

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