Open World Dempster-Shafer Using Complementary Sets

Erik Skau, Cassandra Armstrong, Duc P. Truong, David Gerts, Kari Sentz

International Symposium on Imprecise Probabilities: Theories and Applications, ISIPTA 2023
July 7, 2023

Problem Description

Dempster-Shafer Theory (DST) is a mathematical framework that allows for the preservation of uncertainty and ignorance in decision making. Similar to classical probability, DST suffers from a closed-world assumption, meaning the collection of propositions or hypotheses, called the frame of discernment, cannot be changed.

Complementary Dempster Shafer Theory

Here, we create a modified version of DST with an open-world framework allowing for the insertion of new propositions into the frame of discernment by introducing complementary focal elements.
Complementary focal elements are members of the Cartesian product between the power set of a frame of discernment and the Boolean space, $(u, a) \in 2^{\Omega} \times \mathbb{B}$.

$(u, True)$ represents the hypothesis of u, while $(u, False)$ represents the hypothesis of everything other than u, including propositions not in the current frame of discernment.
Complementary focal elements on one frame of discernment, $(u, a) \in 2^{\Omega_1} \times \mathbb{B}$ can be evaluated to focal elements on a different frame of discernment Ω_2 using the identities

$$\begin{aligned} (u, True)|_{\Omega_2} &= \Omega_2 \cap u \\ (u, False)|_{\Omega_2} &= \Omega_2 - u \; . \end{aligned} \tag{1}$$

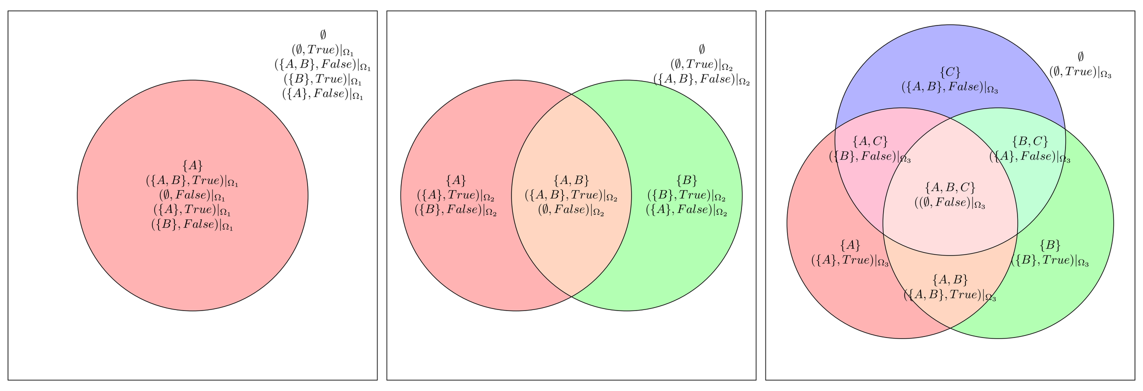


Figure: Examples of DST propositions evaluated on different frames of discernment. From left to right, the frames of discernment are $\Omega_1 = \{A\}$, $\Omega_2 = \{A, B\}$, and $\Omega_3 = \{A, B, C\}$

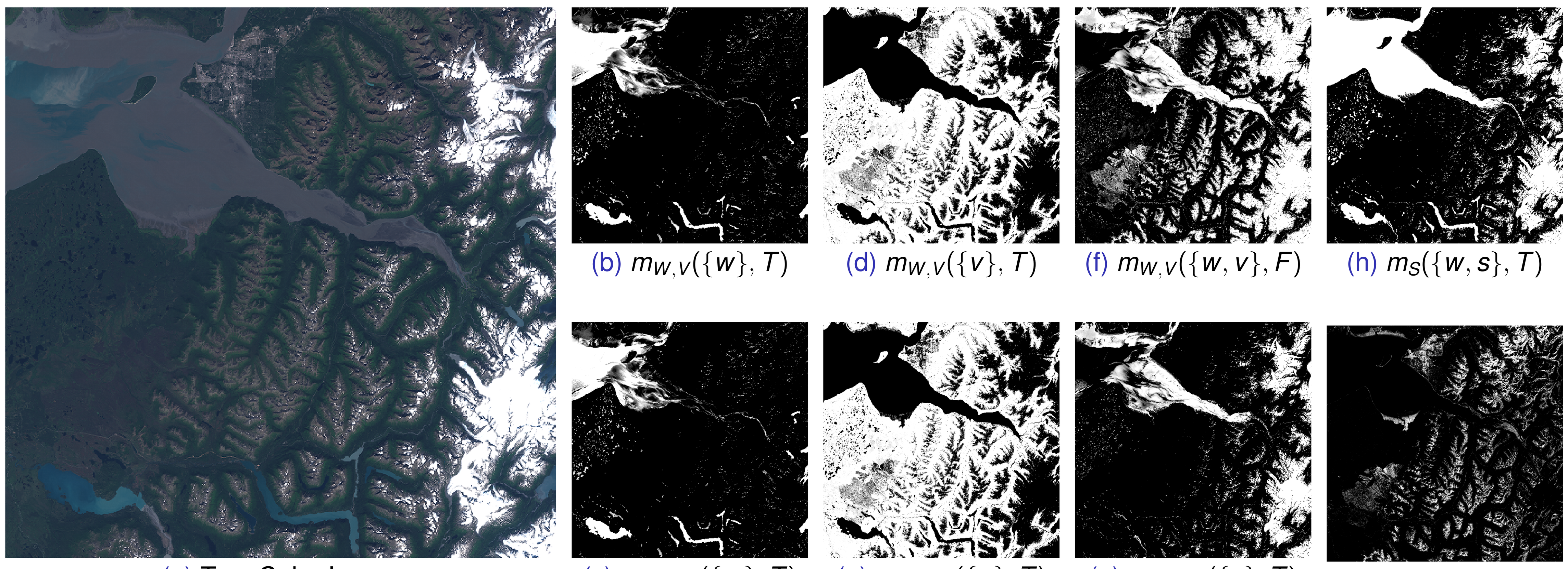
In CDST, we assign mass to focal elements and their complements in Complementary Basic Probability Assignments (CBPA).
A *Complementary Basic Probability Assignment* (CBPA) on a frame of discernment, Ω, is a bivariate mapping

$$m : 2^{\Omega} \times \mathbb{B} \to [0, 1]$$

that satisfies $\sum_{x \in 2^{\Omega} \times \mathbb{B}} m(x) = 1$.

Complementary focal elements can be intersected with one another using the identities

$$\begin{aligned} (u, True) \cap (v, True) &= (u \cap v, True) \\ (u, True) \cap (v, False) &= (u - v, True) \\ (u, False) \cap (v, True) &= (v - u, True) \\ (u, False) \cap (v, False) &= (v \cup u, False) \; . \end{aligned} \tag{2}$$



(a) True Color Image
(b) $m_{W,V}(\{w\}, T)$
(c) $m_{W,V,S}(\{w\}, T)$
(d) $m_{W,V}(\{v\}, T)$
(e) $m_{W,V,S}(\{v\}, T)$
(f) $m_{W,V}(\{w, v\}, F)$
(g) $m_{W,V,S}(\{s\}, T)$
(h) $m_S(\{w, s\}, T)$
(i) $m_{W,V,S}(\{w, v, s\}, F)$

Figure: These images show the true color image of anchorage Alaska with various CBPA mass assignments for water, vegetation, and snow.

Conjunctive Join

The conjunctive join rule provides a procedure for fusing CBPAs together even if they do not share a frame of discernment.
Let m_1 and m_2 be CBPAs with frames of discernment Ω_1 and Ω_2 respectively. The *conjunctive join* $m_{1,2} = m_1 \oplus m_2$ is another CBPA with frame of discernment $\Omega_1 \cup \Omega_2$, defined by

$$m_{1,2} : 2^{\Omega_1 \cup \Omega_2} \times \mathbb{B} \to [0, 1] \tag{3}$$

$$m_{1,2}(z) = \sum_{\substack{x \in 2^{\Omega_1} \times \mathbb{B} \\ y \in 2^{\Omega_2} \times \mathbb{B} \\ x \cap y = z}} m_1(x) \cdot m_2(y) \; . \tag{4}$$

Land Cover Classification

Subject matter experts use normalized differences to determine the presence of specific materials in multi-spectral imagery, typically utilizing two bands from the multi-spectral data to indicate the likelihood that a material exists within a pixel.

$$\mathrm{ND}(X_1, X_2) = \frac{X_1 - X_2}{X_1 + X_2} \tag{5}$$

In this study, we use the normalized difference water index, vegetation index, and snow index to create CBPAs, m_W, m_V, and m_S respectively. We compute CBPA mass assignments as clipped linear transformations of the normalized differences,

$$\begin{aligned} m_{Method}((Materials, True)) &= \frac{\mathrm{ND}(X_1, X_2) - T_l}{T_u - T_l} \\ m_{Method}((Materials, False)) &= 1 - \frac{\mathrm{ND}(X_1, X_2) - T_l}{T_u - T_l} \; . \end{aligned} \tag{6}$$

CBPA	Materials	X_1	X_2	T_l	T_u
m_W	$\{water\}$	B3	B8A	0.3	0.5
m_V	$\{vegetation\}$	B8A	B4	0.2	0.5
m_S	$\{water, snow\}$	B3	B11	-0.1	0.2

Test Data

To demonstrate a use of CDST for land cover classification, we chose to classify a multi-spectral image taken by the Sentinel-2 satellite for the European Space Agency over Anchorage, Alaska. This data includes reflectance data from 13 spectral bands.

Results

From the Anchorage imagery, we can see that the classification improves when we add more elements to our frame of discernment through fusing mass functions with differing frames of discernment. Additionally, we see in the two *false* images that there are far fewer unclassified pixels for the frame of discernment with three elements than that with two. The open world framework allows for elements, such as snow, to be added resulting in a more complete and correctly classified image. The pixels that remain unclassified largely correspond with materials outside the three element frame of discernment, such as urban areas or open rock.
With this work we have demonstrated CDST's dynamic framework provides an ability to quantify missing information. With CDST we can: (1) identify the need for the addition of new propositions to the frame of discernment (or sample space); (2) identify the need for the addition of new classifiers or sensors to identify these new propositions; and (3) reconcile this formulation with an open world.

Acknowledgments

Research presented in this poster was supported by the Laboratory Directed Research and Development program of Los Alamos National Laboratory under project number 20220799DI.