A Nonstandard Approach To Stochastic Processes Under Probability Bounding

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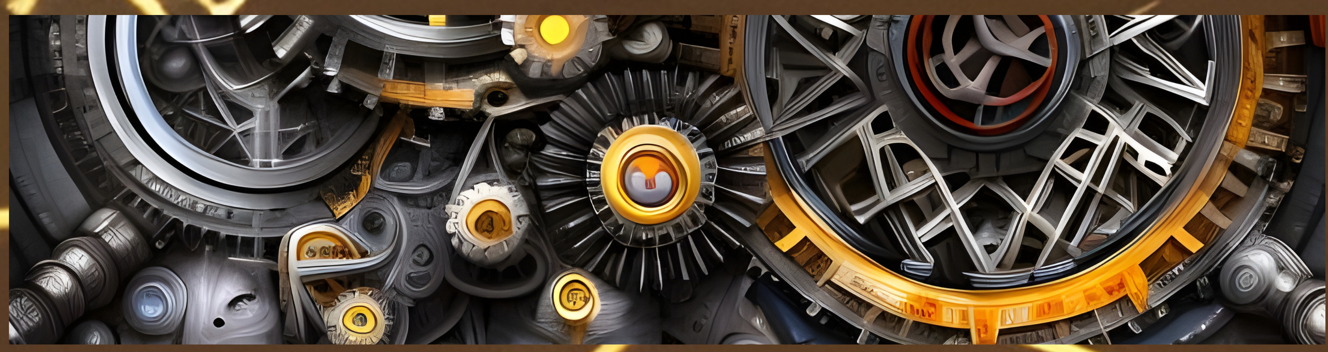


What is internal set theory?

- new predicate ‘**standard**’ that applies to objects (sets, functions, …)
- three new axioms added to ZFC to govern use of this predicate [4]

Intuition?

- internal formulas: do not use ‘standard’   “$0 + 1$ equals 1”
- external formulas: do use ‘standard’   “0 is a standard natural number”
- for an object to be standard, intuitively, we mean [2, §1.1.1, p. 2]:

  ‘at any stage within the mathematical discourse, [...]
  uniquely defined [using an explicitly written internal formula]’

Why is it useful?

- has the notion of an **infinitesimal** so the theory formalizes intuition that goes back to Newton, Leibniz, Cauchy, etc. [1]
- many objects have an infinitely close standard object, called its **shadow**: allows us to move between standard functions with infinite domain and non-standard functions with finite domain

  very useful to study stochastic processes in continuous time!

Difficulties

- for historical reasons, most mathematicians are unfamiliar with it
- application of new axioms needs care: requires some retraining in logic
- illegal set formation: cannot form sets with external formulas

What is a stochastic process?

Notation:

- standard finite state space $\mathcal{X}$
- index set $T_0 \subseteq \mathbb{R}_+$
- possibility space Ω
- function $\xi\colon T_0 \times \Omega \to \mathcal{X}$ given outcome ω at time t, the state is $\xi(t, \omega)$
- $T \subseteq T_0$ any subset of T_0 (e.g. finite discretization)
- $\mathcal{A}(T)$ algebra generated by events of the form $\{\xi(t) = x\}$
- $\mathcal{L}(T)$ linear space spanned by $\mathcal{A}(T)$
- $\mathcal{K}(T) := \mathcal{L}(T) \times (\mathcal{A}(T) \setminus \{\emptyset\})$

Definition

A *(stochastic) process* on T is a coherent lower prevision $\underline{\mathbb{E}}$ defined on $\mathcal{K}(T)$.

Nearby elementary processes

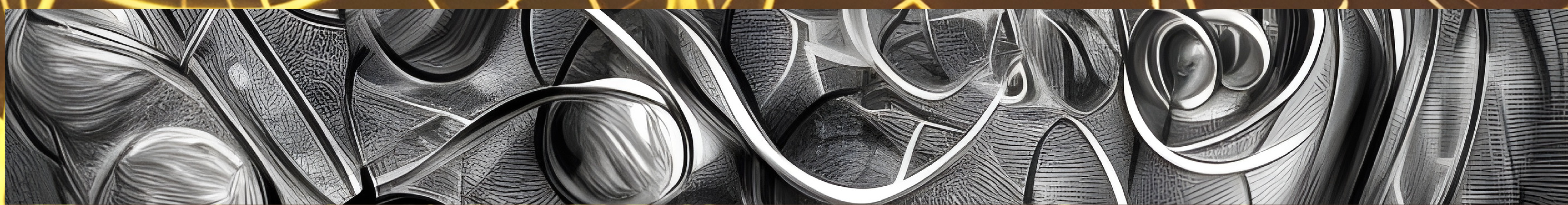
Theorem

Assume $T \subseteq T_0$ and T contains all standard elements of T_0.
Let $\underline{\mathbb{E}}\colon \mathcal{K}(T) \to \mathbb{R}$ be any process.
Then there is a unique standard process $\underline{\mathbb{E}}_0\colon \mathcal{K}(T_0) \to \mathbb{R}$,
*called the **shadow** of $\underline{\mathbb{E}}$, satisfying*

$$\forall^{\mathrm{s}}(f, A) \in \mathcal{K}(T_0)\colon \underline{\mathbb{E}}(f \mid A) \simeq \underline{\mathbb{E}}_0(f \mid A) \tag{1}$$

Definition

A standard process $\underline{\mathbb{E}}_0$ on T_0 is said to be *nearby* an elementary process $\underline{\mathbb{E}}$ on T if $T \subseteq T_0$, T contains all standard elements of T_0, and $\underline{\mathbb{E}}_0$ is the shadow of $\underline{\mathbb{E}}$.



Imprecise Markov chains: Notation

- T is a finite subset of T_0 containing all standard elements of T_0
- $T' := T \setminus \{\max T\}$
- if $t \in T'$ then $t + dt$ denotes the successor of t in T, i.e.

$$dt := \min\{t' \in T\colon t' > t\} - t \tag{2}$$

- for any function ϕ on T, $\phi(0 : t)$ denotes the restriction of ϕ to $[0, t] \cap T$

Imprecise Markov chains: Definition

- $\underline{\mathbb{I}}$: coherent lower prevision on $\mathbb{R}^{\mathcal{X}}$
- $\underline{\mathbb{T}}_t(\cdot)(x)$: coherent lower prevision on $\mathbb{R}^{\mathcal{X}}$ (for each $t \in T'$ and $x \in \mathcal{X}$)

Definition

A precise elementary process $\mathbb{E}$ on T is **compatible** with $(\underline{\mathbb{I}}, \underline{\mathbb{T}})$ if for all paths $x\colon T' \to \mathcal{X}$, all $t \in T'$, and all $f \in \mathbb{R}^{\mathcal{X}}$,

$$\mathbb{E}(f(\xi(0)) \ge \underline{\mathbb{I}}(f) \tag{3}$$

$$\mathbb{E}(f(\xi(t + dt)) \mid \xi(0 : t) = x(0 : t)) \ge \underline{\mathbb{T}}_t(f)(x(t)) \tag{4}$$

If $\underline{\mathbb{E}}$ denotes the lower envelope of all these compatible precise elementary processes, then $\underline{\mathbb{E}}$ is called the **elementary imprecise Markov chain** induced by $(\underline{\mathbb{I}}, \underline{\mathbb{T}})$. Its shadow is called the **imprecise Markov chain** induced by $(\underline{\mathbb{I}}, \underline{\mathbb{T}})$.

Continuous time imprecise Markov chains

- fix a standard **lower rate operator** $\underline{\mathbb{Q}}\colon \mathbb{R}^{\mathcal{X}} \to \mathbb{R}^{\mathcal{X}}$ and define

$$\underline{\mathbb{T}}_t := I + dt\underline{\mathbb{Q}} \tag{5}$$

- the shadow $\underline{\mathbb{E}}_0$ of this elementary imprecise Markov chain behaves just like the usual continuous time imprecise Markov chains

Theorem

For all $x \in \mathcal{X}$, $f \in \mathbb{R}^{\mathcal{X}}$, and $t \ge 0$, $\underline{\mathbb{E}}_0(f(\xi(t)) \mid \xi(0) = x) = e^{t\underline{\mathbb{Q}}}(f)(x)$.

Sketch of proof

Assume $t > 0$. By transfer, only need to establish the equality for standard t.
It suffices to show that

$$\left\| \prod_{s<t} (I + ds\underline{\mathbb{Q}}) - e^{t\underline{\mathbb{Q}}} \right\| \simeq 0 \tag{6}$$

Indeed, with $\delta := \max_{s<t} ds$,

$$\left\| \prod_{s<t} (I + ds\underline{\mathbb{Q}}) - e^{t\underline{\mathbb{Q}}} \right\| = \left\| \prod_{s<t} (I + ds\underline{\mathbb{Q}}) - \prod_{s<t} e^{ds\underline{\mathbb{Q}}} \right\| \tag{7}$$

$$\le \sum_{s<t} \left\| I + ds\underline{\mathbb{Q}} - e^{ds\underline{\mathbb{Q}}} \right\| \lesssim \sum_{s<t} (ds)^2 \|\underline{\mathbb{Q}}\|^2 \tag{8}$$

$$\le \sum_{s<t} (ds)\delta\|\underline{\mathbb{Q}}\|^2 = t\delta\|\underline{\mathbb{Q}}\|^2 \simeq 0 \tag{9}$$

since t and $\|\underline{\mathbb{Q}}\|$ are limited, and $\delta \simeq 0$.   (compare with [3])

[1] Alexandre Borovik and Mikhail G. Katz, *Who gave you the Cauchy–Weierstrass tale? The dual history of rigorous calculus*, Foundations of Science **17** (2012), no. 3, 245–276.

[2] Francine Diener and Marc Diener (eds.), *Nonstandard analysis in practice*, Springer, 1995.

[3] Thomas Krak, Jasper De Bock, and Arno Siebes, *Imprecise continuous-time Markov chains*, International Journal of Approximate Reasoning **88** (2017), 452–528.

[4] Edward Nelson, *Radically elementary probability theory*, Annals of Mathematical Studies, Princeton University Press, New Jersey, 1987.