Optimization Problems with Evidential Linear Objective

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Context

This work is centered around an optimization problem with a linear objective (LOP):

$$\max\ c^T x \quad \text{s.t. } x \in \mathcal{X} \subseteq \mathbb{Z}^{n_1}_{\geq 0} \times \mathbb{R}^{n_2}_{\geq 0} \text{ with } n_1 + n_2 = n. \tag{LOP}$$

- In real life, coefficients c_i in LOP are usually uncertain.
- Uncertainty can be represented by discrete scenario sets (*e.g.*,$\{\boldsymbol{c_1}, \boldsymbol{c_2}\}$) or by intervals:

(a) Scenario $c_1 = (2, 5, 4, 3, 4)$ (b) Scenario $c_2 = (3, 4, 5, 1, 2)$

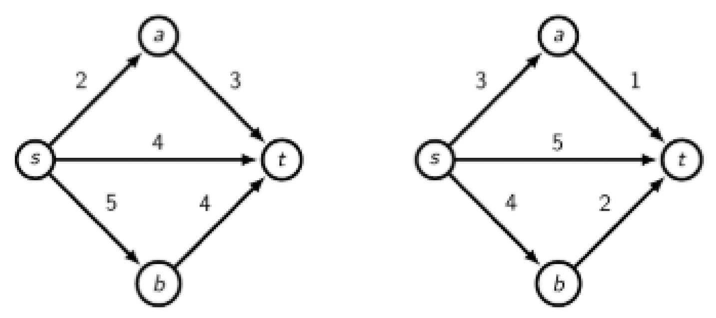


Figure: Discrete scenario representation

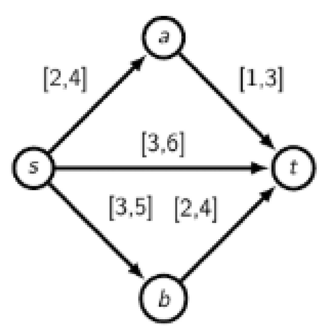
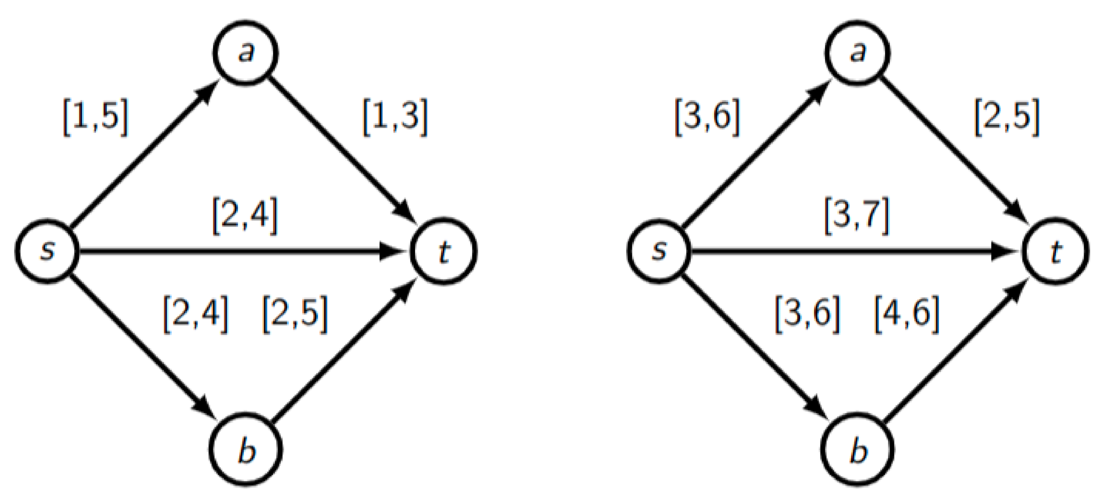


Figure: Interval representation

Our focus

- We investigate the case where the uncertainty on the coefficients c_i of c is *evidential*, *i.e.*, modelled by a belief function.
- We consider the case where focal sets of the belief function are Cartesian product of compact sets, with each compact set describing possible values of each coefficient.
- Such a belief function is a direct and natural generalization of the interval representation found in robust optimization.

(a) Focal set F_1 with $m(F_1) = 0.8$ (b) Focal set F_2 with $m(F_2) = 0.2$



Belief Function Theory

- A frame Ω (closed subset of $\mathbb{R}^n$) contains all possible values of a variable of interest ω.
- We would like to quantify the uncertainty about statements like "the subset A of Ω contains the true value of ω".
- It can be represented by a mapping $m : \mathcal{C} \mapsto [0, 1]$ called mass function, where $\mathcal{C}$ is assumed here to be a finite collection of closed subsets of Ω, such that:
$$\sum_{A \in \mathcal{C}} m(A) = 1 \text{ and } m(\emptyset) = 0$$
- $A \subseteq \Omega$ is called focal set if $m(A) > 0$. The set of all focal sets of m is denoted by $\mathcal{F}$.
- m induces a belief function Bel and a plausibility function Pl defined on $\mathcal{B}(\Omega)$ the Borel subsets of Ω:
$$Bel(A) = \sum_{B \in \mathcal{F}: B \subseteq A} m(B), \quad Pl(A) = \sum_{B \in \mathcal{F}: B \cap A \neq \emptyset} m(B).$$
- A probability measure P on $\mathcal{B}(\Omega)$ is compatible with m if $Bel(A) \leq P(A)\ \forall A \in \mathcal{B}(\Omega)$. Let $\mathcal{P}(m)$ be set of all compatible probability measures.

Evidential coefficients in the objective

- Let Ω_i be the set of possible values for c_i and let $\Omega := \times_{i=1}^n \Omega_i$. Each $c \in \Omega$ is called a scenario.
- A mass function m on Ω, with set of focal sets $\mathcal{F} = \{F_1, \dots, F_K\}$, represents uncertainty about the coefficients.
- We assume that each focal set is a Cartesian product of compact sets, i.e., $F_r = \times_{i=1}^n F_r^{\downarrow i},\ \forall r \in \{1, \dots, K\}$.
- The minimum and maximum values of $F_r^{\downarrow i}$ are denoted by l_i^r and u_i^r, respectively.

Expected values of solutions

- Each $x \in \mathcal{X}$ can be viewed as an act $x : \Omega \to \mathbb{R}$ such that $x(c) = \sum_{i=1}^n x_i c_i$.
- The *upper expected value* $\overline{E}_m(x)$ and *lower expected value* $\underline{E}_m(x)$ of x, relative to m, are defined as:
$$\overline{E}_m(x) := \sup_{P \in \mathcal{P}(m)} E_P(x), \quad \underline{E}_m(x) := \inf_{P \in \mathcal{P}(m)} E_P(x).$$
- It can be shown that
$$\overline{E}(x) = \sum_{i=1}^n \bar{u}_i x_i,\ \underline{E}(x) = \sum_{i=1}^n \bar{l}_i x_i,$$
with $\bar{u}_i := \sum_{r=1}^K m(F_r) u_i^r$ and $\bar{l}_i := \sum_{r=1}^K m(F_r) l_i^r$, $i = 1, \dots, n$.

Possibly optimal solutions

- This notion appears in many works in minimax regret optimizations with interval data.
- x is a possibly optimal solution of LOP with respect to the set $\mathcal{C} := \times_{i=1}^n [\bar{l}_i, \bar{u}_i]$ if x is an optimal solution for at least one c in $\mathcal{C}$.
- Let $Opt_{pos}^{\mathcal{C}}$ be the set of all possibly optimal solutions.

Comparing solutions

- With criteria in decision making under uncertainty, we can compare solutions as:
  - Generalized Hurwicz: $x \succeq_{hu}^{\alpha} y$ if
  $$\alpha \overline{E}_m(x) + (1-\alpha)\underline{E}_m(x) \geq \alpha \overline{E}_m(y) + (1-\alpha)\underline{E}_m(y),$$
  for some fixed $\alpha \in [0, 1]$.
  - Strong dominance: $x \succeq_{str} y$ if
  $$\underline{E}_m(x) \geq \overline{E}_m(y).$$
  - Weak dominance: $x \succeq_{weak} y$ if
  $$\overline{E}_m(x) \geq \overline{E}_m(y) \text{ and } \underline{E}_m(x) \geq \underline{E}_m(y).$$
  - Maximality: $f \succeq_{max} g$ if
  $$\underline{E}_m(x - y) \geq 0 \Leftrightarrow \forall P \in \mathcal{P}(m), E_P(x) \geq E_P(y).$$
  - E-admissibility: x is E-admissible if $\exists P \in \mathcal{P}(m)$ such that $E_P(x) \geq E_P(y)\ \ \forall y$.

Best solutions definitions

For $\succeq_{cr}$ in $\{\succeq_{hu}^{\alpha}, \succeq_{str}, \succeq_{weak}, \succeq_{max}, \succeq_{adm}\}$, the set of non-dominated (best) solutions:

$$Opt_{cr} = \{x : \nexists y \text{ such that } y \succ_{cr} x\}.$$

Results

We characterize non-dominated solutions by established concepts in optimization.

- For any LOP,
  - Solutions in Opt_{hu}^{α} are characterized in terms of optimal solutions of LOP with coefficients $c_i = \alpha\bar{u}_i + (1-\alpha)\bar{l}_i$.
  - Solutions in Opt_{str} are characterized in terms of solutions of a *lower-bound feasibility problem* associated with LOP.
  - Solutions in Opt_{weak} are characterized in terms of *efficient* solutions of a bi-objective LOP, where each c_i has two weights $\bar{u}_i, \bar{l}_i$.
  - $Opt_{adm} \subseteq Opt_{pos}^{\mathcal{C}} \subseteq Opt_{max}$.
- Moreover,
  - If LOP is a linear mixed-integer programming (*i.e.*, $\mathcal{X}$ is in the form of $Mx \leq b$ for a matrix M and a vector b), $Opt_{adm} = Opt_{pos}^{\mathcal{C}}$ and in general $Opt_{pos}^{\mathcal{C}} \subset Opt_{max}$.
  - If LOP is convex (*i.e.*, $\mathcal{X}$ is convex) or combinatorial (*i.e.*, $\mathcal{X} \subseteq \{0, 1\}^n$), $Opt_{adm} = Opt_{pos}^{\mathcal{C}} = Opt_{max}$.

Important consequences

- Finding non-dominated solutions amounts to solving the deterministic version or well-known variants of the problem. Hence, fast existing methods can be applied.
- As α varies from 0 to 1, solutions in Opt_{hu}^{α} can be found at once by using method from parametric optimizations such as parametric simplex method.
- In general, checking E-admissibility is costly. But thanks to the link with $Opt_{pos}^{\mathcal{C}}$, it implies that:
  - In the case of linear programming (number of acts is infinite), checking E-admissibility is fast by just using simplex algorithms.
  - In the case of combinatorial optimization problem (number of acts is finite but extremely huge), if the deterministic problem can be solved efficiently (*e.g.*, the shortest path problem), checking E-admissibility is also efficient.
- Open problem: $\exists$ an instance of LOP such that $Opt_{adm} \subset Opt_{pos}^{\mathcal{C}}$?

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